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THE UNIVERSITY OF ALBERTA
SOME OSCILLATION AND NONOSCILLATION RESULTS FOR
GENERALIZED EMDEN-FOWLER EQUATION $(r(x)y')' + p(x)y^7 = 0$

by

 Hong Lu

A THESIS SUBMITTED TO
THE FACULTY OF GRADUATE STUDIES AND RESEARCH
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled Some Oscillation and Nonoscillation Results for The Generalized Emden-Fowler Equation $(r(x)y')' + p(x)y^7 = 0$ submitted by Hong Lu in partial fulfilment of the requirements for the degree of Master of Science.

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D. S. W. +

Date

Abstract

We obtain some new non-oscillation results for the generalized Emden-Fowler differential equation

$$(r(x)y')' + p(x)y^\gamma = 0, \quad \gamma > 0.$$

The techniques involve a change of variable and analysis of related energy functions.

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I. INTRODUCTION

The second order nonlinear ordinary differential equation

$$(1.1) \quad (t^p x')' + t^\sigma x^\gamma = 0, \quad t \geq 0$$

where p and σ are real numbers and γ is the quotient of two positive odd integers, is referred to as the Emden-Fowler equation in the literature. By the generalized Emden-Fowler equation we mean the following natural generalization of (1.1):

$$(1.2) \quad (r(t)x')' + p(t)x^\gamma = 0, \quad t \geq a$$

where $r(t)$ is absolutely continuous and positive, $p(t)$ is continuous and positive, and γ is as above.

The study of equation (1.1) originates from problems concerning gaseous dynamics in astrophysics. Lane [34] studied the so-called Lane-Emden equation

$$(1.3) \quad (t^2 x')' + t^2 x^n = 0.$$

For some special values of n , we can solve (1.3) explicitly. For example, when $n = 0$,

$$x(t) = \frac{c_1}{t} + c_2 - \frac{t^2}{6}$$

and when $n = 1$

$$x(t) = \frac{1}{t}(c_1 \cos t + c_2 \sin t).$$

The physical aspects of equation (1.3) are summarized in Emden [16]. The mathematical foundation for the study of such an equation and also of the general

equation (1.1) was made by Fowler in a series of four papers [21]-[24]. The basic theory is contained in Bellman's book [6].

Atkinson [1] - [5] made some of the first investigations on the generalized Emden-Fowler equation (1.2) for $r(t) \equiv 1$, i.e., the equation

$$(1.4) \quad x'' + p(t)x^\gamma = 0, \quad t \geq a.$$

His results have become standard in the following years. Many of them are contained in Cesari [12] and Coppel [15]. There are also many survey articles such as Jackson [28], Willett [34] and Wong [44].

Except for some very special cases as we have seen, it is impossible to find explicit solutions for the equation (1.1), (1.2) or (1.4). Thus it is very important both mathematically and physically to understand the asymptotic behavior of solutions of these equations. Among various qualitative properties, the oscillatory behavior of the solutions is of particular importance.

A nontrivial solution $x(t)$ of equation (1.2) is called oscillatory if it has infinitely many zeros in $[a, \infty)$. Otherwise it is called nonoscillatory. Equation (1.2) is said to be oscillatory if all its solutions are oscillatory and similarly, it is said to be nonoscillatory if all its nontrivial solutions are nonoscillatory.

When the equation is linear, i.e., $\gamma \equiv 1$, then by the Sturm-Liouville theory equation (1.2) is either oscillatory or nonoscillatory. But in the nonlinear case equation (1.2) or even equation (1.1) may have both oscillatory and nonoscillatory

solutions. Fowler [24] proved that when $\gamma > 1$ and $\sigma + 2 < 0 \leq \sigma + \frac{\gamma+3}{2}$, then the equation

$$(1.5) \quad x'' + t^\sigma x^\gamma = 0$$

has both oscillatory and nonoscillatory solutions. Fowler [24] obtained the complete classification of the oscillatory behavior of (1.5).

For the generalized Emden-Fowler equation (1.4), Atkinson [3] showed that when $\gamma > 1$ (1.4) is oscillatory if and only if

$$\int_a^\infty t p(t) dt = \infty.$$

When $0 < \gamma < 1$, Belohovec [8] proved that (1.4) is oscillatory if and only if

$$\int_a^\infty t^\gamma p(t) dt = \infty.$$

Although various conditions are obtained for equation (1.2) to be nonoscillatory, the classification of (1.2) is far from being completed. Many criteria are surveyed in Willett [39], Wong [41] and [44] and Erbe [18].

In this thesis we first give a summary of the known results concerning the oscillatory behavior of the solutions of equation (1.2). Because of Atkinson's results, most efforts were made toward various nonoscillatory criteria. Also since the function factor $r(t)$ in (1.2) can be eliminated via some standard changes of variables, most authors considered only the special case that $r(t) \equiv 1$. This survey comprises Chapter II.

In Chapter III we approach the oscillation problem in a different way. Instead of dealing with the special case that $r(t) \equiv 1$, we consider directly the general equation (1.2) itself. We make use of techniques developed for the special case and obtain a number of nonoscillation criteria for (1.2). Although our results are all generalizations and improvements of theorems we state in Chapter II, most of them can not be obtained directly by the standard changes of variables.

II. SUMMARY FOR THE OSCILLATORY BEHAVIOR

OF THE EQUATION $x'' + p(t)x^\gamma = 0$

In this chapter we shall survey the known results on the oscillatory behavior of the special case of equation (1.2) that $r(t)$ is constant 1, that is the oscillation of the equation

$$(2.1) \quad x'' + p(x)x^\gamma = 0, \quad t \geq a$$

where $p(t)$ is continuous and positive, γ is the quotient of two positive odd integers.

Equation (2.1) is of particular importance since the more generalized equation

$$(2.2) \quad (r(t)x')' + p(t)x^\gamma = 0, \quad t \geq a$$

can be reduced to the form (2.1) via a standard change of variables. To see this, suppose first that $\int_a^\infty \frac{dt}{r(t)} = \infty$, then the Liouville transformation

$$(2.3) \quad s = \int_a^t \frac{d\tau}{r(\tau)}, \quad y(s) = x(t)$$

reduces (2.2) to

$$y'' + r(t)p(t)y^\gamma = 0, \quad s \geq 0.$$

On the other hand, when $\int_a^\infty \frac{dt}{r(t)} < \infty$, the transformation

$$(2.4) \quad s = \left(\int_t^\infty \frac{d\tau}{r(\tau)} \right)^{-1}, \quad y(s) = s \cdot x(t)$$

gives us the equivalent equation for (2.2) in new variables:

$$y'' + \frac{r(t)p(t)}{s^{\gamma+3}} y^{\gamma}(s) = 0, \quad s \geq \left(\int_a^{\infty} \frac{d\tau}{r(\tau)} \right)^{-1}.$$

§1 THE LINEAR EQUATION

In this section we concentrate on the case $\gamma = 1$, i.e., the equation

$$(2.5) \quad x'' + p(t)x = 0.$$

We have the following powerful tool:

Sturm separation Theorem: Let x_1 and x_2 be two linearly independent solutions of (2.5). If there exists $t_1 < t_2$ such that $x_1(t_1) = x_1(t_2) = 0$, then $x_2(t)$ has a zero in the interval (t_1, t_2) .

We obtain from the Sturm separation Theorem that if there exists an oscillatory solution of (2.5), then every solution is oscillatory. Thus if (2.5) is not nonoscillatory, then it is oscillatory.

Kneser [32] gave one of the first criteria for the oscillation of (2.5). He proved that if the limit $w = \lim_{t \rightarrow \infty} t^2 p(t)$ exists, then (2.5) is oscillatory if $w > \frac{1}{4}$ and nonoscillatory if $w < \frac{1}{4}$.

Hille [27] generalized Kneser's results to obtain his famous theorem.

THEOREM 2.1 (HILLE). (i) Equation (2.5) is oscillatory if

$$\liminf_{t \rightarrow \infty} t \int_t^{\infty} p(\tau) d\tau > \frac{1}{4};$$

(ii) Equation (2.5) is nonoscillatory if

$$\limsup_{t \rightarrow \infty} t \int_t^{\infty} p(\tau) d\tau < \frac{1}{4}.$$

As a corollary, we see that (2.5) is oscillatory if $\int_a^{\infty} p(t) dt = \infty$.

Hille's theorem still remains as one of the most important linear oscillation criteria. Since then, many refinements of Theorem 2.1 have been obtained and some necessary and sufficient conditions have been established. But none of them is as simple and applicable as Theorem 2.1. The following are necessary and sufficient conditions obtained by Nehari [36] and Hille [27].

THEOREM 2.2 (NEHARI). Consider the eigenvalue problem

$$x'' + \lambda p(t)x = 0, \quad x(\alpha) = x'(\beta) = 0.$$

Let λ be the smallest eigenvalue. Then (2.5) is nonoscillatory if and only if $\lambda > 1$ for all $\beta > \alpha$.

THEOREM 2.3 (HILLE). Equation (2.5) is nonoscillatory if and only if the integral equation

$$\nu(t) = \int_t^{\infty} \nu^2(\tau) d\tau + \int_t^{\infty} p(\tau) d\tau$$

has a solution for sufficiently large t .

We now state oscillation criteria of Bocher [9], Wintner [40] and Leighton [35]. More necessary or sufficient oscillation or nonoscillation conditions can be found in Swanson [38].

THEOREM 2.4 (BOCHER). If $\int_a^\infty tp(t)dt < \infty$, the equation (2.5) is nonoscillatory.

THEOREM 2.5 (LEIGHTON). If p is nonincreasing and $\int_a^\infty (p(t))^{1/2} dt < \infty$, then equation (2.5) is nonoscillatory.

THEOREM 2.6 (WINTNER). Equation (2.5) is oscillatory if

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \int_0^t p(\tau) d\tau dt = \infty.$$

Leighton [35] also considered the same general equation

$$(2.6) \quad (r(t)x')' + p(t)x(t) = 0, \quad t \geq a.$$

He proved the following:

THEOREM 2.7 (LEIGHTON). Suppose $r(t)$ is positive on $[a, \infty)$, if

$$\int_a^\infty \frac{dt}{r(t)} = \int_a^\infty p(t)dt = \infty,$$

then equation (2.6) is oscillatory.

§2 THE SUPERLINEAR EQUATIONS

Equation (2.1) and (2.2) are called superlinear if $\gamma > 1$, they are called sublinear if $0 < \gamma < 1$. They are both nonlinear equations. In this case the oscillatory classification is not as simple as the linear case as we mentioned before. But we do have some better criteria than those for linear equations. In this section we give some oscillation conditions for the superlinear equation

$$(2.7) \quad x'' + p(t)x^\gamma = 0, \quad \gamma > 1.$$

The following section is devoted to the superlinear equation.

Fowler [24] considered the special case that $p(t) = t^\sigma$, where σ is a real number, i.e., equation (1.5). He proved the following results.

- (i) if $\sigma + 2 \geq 0$, then (1.5) is oscillatory;
- (ii) if $\sigma + \frac{\gamma+3}{2} < 0$, then (1.5) is nonoscillatory;
- (iii) if $\sigma + 2 < 0 \leq \sigma + \frac{\gamma+3}{2}$, then (1.5) has both oscillatory and nonoscillatory solutions.

The earliest oscillation condition for (2.7) is given in Atkinson [3].

THEOREM 2.8 (ATKINSON). *Equation (2.7) is oscillatory if and only if*

$$\int_a^\infty tp(t)dt = \infty.$$

If we compare with Theorem 2.1 we will find Atkinson's beautiful result does not hold for linear equations, though Bocher's result shows that the "only if" part is correct.

From Fowler's result we see that a superlinear equation which is not oscillatory is not necessarily nonoscillatory. Also we see that the fraction $\frac{\gamma+3}{2}$ could play an important role in nonoscillation criteria for equation (2.7). The following two theorems represent early efforts in this directions.

In the following, we always denote $t^{\frac{\gamma+3}{2}}p(t) = \varphi(t)$.

THEOREM 2.9 (JASNY [27] AND KURZWEIL [33]). *If $\varphi(t)$ is nondecreasing, then (2.7) has an oscillatory solution.*

THEOREM 2.10 (KIGURADZE [31]). *If $t^\epsilon \varphi(t)$ is nonincreasing for some $\epsilon > 0$, then (2.7) is nonoscillatory.*

Nehari [37] introduced a logarithm factor into this type of condition. He proved the following theorem.

THEOREM 2.11 (NEHARI). *If $(\log(t))^{\frac{\gamma+3}{2}}\varphi(t)$ is nonincreasing then (2.7) is nonoscillatory.*

Both Theorems 2.9 and 2.11 were improved by Erbe and Muldowney [19] and [20] to the following theorems.

THEOREM 2.12 ([19]). *If $(\log t)^\epsilon \varphi(t) = \phi(t) - \nu(t)$, where $\epsilon \leq 0$, ϕ, ν are non-decreasing and $\phi(t) - \nu(\infty) > 0$ for some t , then (2.7) has an oscillatory solution.*

THEOREM 2.13 ([20]). *If there are real numbers β and η such that $\beta > 0$, $\frac{\beta}{\gamma+3} + \eta(\gamma-1) \geq \frac{1}{2}$, $(\log t)^\eta \varphi(t)$ is nonincreasing and $(\log t)^\beta \varphi(t)$ is bounded, then (2.7) is nonoscillatory.*

In particular, Theorem 2.13 implies Chiou's result in [4] that if $(\log t)^\beta \varphi(t)$ is nonincreasing, where $\beta > \frac{\gamma+1}{4} - \frac{1}{\gamma+1}$, then (2.7) is nonoscillatory.

Atkinson [3] also considered nonoscillation criterion for equation (2.7) with the condition that $p(t)$ is nonincreasing. He proved that if $p(t)$ is nonincreasing and satisfies $\int_0^\infty t^\gamma p(t) < \infty$, then (2.7) is nonoscillatory.

Gouwitz [25] replaced the monotone assumption on $p(t)$ by the weaker condition that $p(t)$ is locally of bounded variation and satisfies

$$(2.8) \quad \int_a^\infty \frac{dp_+(t)}{p(t)} < \infty,$$

where $p(t) = p_+(t) - p_-(t)$ is the Jordan decomposition. He proved a nonoscillation criterion which was generalized by Wong [43] to the following:

THEOREM 2.14 (WONG). Let $p(t)$ be locally of bounded variation and satisfy (2.8). If $\lim_{t \rightarrow \infty} t^{\gamma+1} p(t) = 0$, then (2.7) is nonoscillatory.

Erbe [17] introduced the functions

$$(2.9) \quad \begin{aligned} Q_+(t) &= \exp\left(\int_a^t \frac{dp_+(t)}{p(t)}\right), \quad a \leq t < \infty \\ Q_-(t) &= \exp\left(\int_a^t \frac{dp_-(t)}{p(t)}\right), \quad a \leq t < \infty \end{aligned}$$

and made several extensions of Atkinson's, Gollwitzer's and Wong's results. In fact, his following theorem is also valid for $\gamma = 1$, i.e., for the linear equation (2.5).

THEOREM 2.15 (ERBE). Equation (2.7) is nonoscillatory in case any of the following hold:

- (a) $\int_a^\infty t^{\gamma-1} p(t) (Q_+(t))^{\frac{\gamma-1}{\gamma}} dt < \infty$ and $\lim_{t \rightarrow \infty} t \int_t^\infty r^{\gamma-1} p(r) (Q_+(r))^{\gamma-1} dr = 0$;
- (b) $\int_a^\infty (p(t))^{\frac{2}{\gamma+1}} (Q_+(t))^{\frac{2}{\gamma+1}} dt < \infty$ and $\lim_{t \rightarrow \infty} t \int_t^\infty (p(r))^{\frac{2}{\gamma+1}} (Q_+(r))^{\frac{2}{\gamma+1}} dr = 0$;
- (c) $\int_a^\infty p(t) (Q_-(t))^{\frac{2}{\gamma+1}} dt < \infty$ and $\lim_{t \rightarrow \infty} t \int_t^\infty p(r) (Q_-(r))^{\frac{2}{\gamma+1}} dr = 0$;
- (d) $\int_a^\infty t^\gamma p(t) dt < \infty$ and $\lim_{t \rightarrow \infty} (Q_+(t))^{\frac{\gamma-1}{\gamma}} \int_t^\infty r^\gamma p(r) dr = 0$;
- (e) $\int_a^\infty (p(t))^{\frac{1}{\gamma+1}} (Q_+(t))^{\frac{1}{\gamma+1}} dt < \infty$;
- (f) $\lim_{t \rightarrow \infty} t^{\gamma+1} p(t) (Q_+(t))^{\frac{\gamma-1}{\gamma}} = 0$;
- (g) $\lim_{t \rightarrow \infty} t^2 p(t) (Q_-(t))^{\frac{2}{\gamma+1}} = 0$.

§3 THE SUBLINEAR EQUATIONS

In this section we consider the dual equation of (2.7) the sublinear equation:

$$(2.10) \quad x'' + p(t)x^\gamma = 0, \quad 0 < \gamma < 1.$$

Many theorems we stated in the previous section have their dual forms for equation (2.10). In some sense, sublinear equations are related to superlinear equations more than to linear equations.

Unlike the superlinear equation (2.7), sublinear equations are not very much of physical interest. The reason for studying equation (2.10) is largely mathematical. The analogues of theorems 2.8 and 2.10 were given by Belohorec in [7] and [8].

THEOREM 2.16 (BELOHOREC). *Equation (2.10) is oscillatory if and only if*

$$\int_0^{\infty} t^{\gamma} p(t) dt = \infty.$$

THEOREM 2.17 (BELOHOREC). *Equation (2.10) is nonoscillatory if $t^{\delta} \varphi(t)$ is nondecreasing and bounded above, where $0 < \delta < \frac{1-\gamma}{2}$.*

An analogue of Theorem 2.9 is given in Chiou [13].

THEOREM 2.18 (CHIOU). *If $\varphi(t)$ is nondecreasing, then (2.10) has an oscillatory solution.*

It is interesting to compare the assumptions of Theorem 2.17 and 2.18. They look very similar, but are in fact completely different.

Erbe and Muldowney [19] also gave a logarithm nonoscillation criterion for sublinear equations which is an analogue of Theorem 2.12.

THEOREM 2.19. *If $(\log t)^\epsilon \varphi(t) = \pi(t) - \nu(t)$, where $\epsilon \leq 0$, π, ν are nondecreasing and $\pi(t) - \nu(\infty) > 0$ for some t , then (2.10) has an oscillatory solution.*

As corollaries, we have that

- (i) If $(\log t)^\epsilon \varphi(t)$ is nondecreasing, then (2.10) has an oscillatory solution.
- (ii) If $(\log t)^\epsilon \varphi(t)$ is nonincreasing and bounded below, then (2.10) has an oscillatory solution.

Condition (2.8) was also used in establishing nonoscillation criteria for Equation (2.10). Corresponding to Theorem 2.14 we have the following result of Wong [43].

THEOREM 2.20 (WONG [43]). *Let $p(t)$ be locally of bounded variation and satisfy (2.8). If $\lim_{t \rightarrow \infty} t^2 p(t) = 0$, then (2.10) is nonoscillatory.*

The functions $Q_+(t)$ and $Q_-(t)$ defined in (2.9) are also used in Erbe [17] to obtain nonoscillation criteria for the sublinear equation (2.10).

THEOREM 2.21 (ERBE). *Equation (2.10) is nonoscillatory in case any of the following hold:*

- (a) $\int_a^\infty t^\gamma p(t) dt < \infty$ and $\lim_{t \rightarrow \infty} (p(t))^{1/\gamma} (Q_+(t))^{1/\gamma} \int_t^\infty r^\gamma p(r) dr = 0$;
- (b) $\int_a^\infty t^\gamma p(t) dt < \infty$ and $\lim_{t \rightarrow \infty} (Q_-(t))^{1/\gamma} \int_t^\infty r^\gamma p(r) dr = 0$;
- (c) $\int_a^\infty (p(t))^{1/\gamma} (Q_+(t))^{1/\gamma} dt < \infty$ and $\lim_{t \rightarrow \infty} (Q_-(t))^{1/\gamma} (Q_+(t))^{1/\gamma} \int_t^\infty (p(r))^{1/\gamma} (Q(r))^{1/\gamma} dr = 0$;
- (d) $\int_a^\infty (p(t))^{1/\gamma} (Q_+(t))^{1/\gamma} dt < \infty$ and $\lim_{t \rightarrow \infty} (p(t))^{1/\gamma} (Q_+(t))^{1/\gamma} \int_t^\infty (p(r))^{1/\gamma} (Q_+(r))^{\gamma(\gamma+1)} dr = 0$;
- (e) $\lim_{t \rightarrow \infty} t^2 p(t) (Q_+(t))^{1/\gamma} = 0$;

$$(f) \lim_{t \rightarrow \infty} t^{\gamma+1} p(t) (Q_-(t))^{\frac{1-\gamma}{2}} = 0.$$

We refer to Erbe [18] for the interrelations among Theorem 2.21 and other nonoscillation conditions.

Another class of oscillation criteria is the integral averaging conditions. The first of them is Wintner's theorem (Theorem 2.6). His result remains valid for the superlinear equation (2.7) as proved in Butler [10].

THEOREM 2.22 (BUTLER). *The superlinear equation (2.7) is oscillatory if*

$$(2.11) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_a^T \int_a^t p(\tau) d\tau dt = \infty.$$

In the sublinear case, the same result is also true. Kamenev [30] proved a stronger result.

THEOREM 2.23 (KAMENEV). *The sublinear equation (2.10) is oscillatory if*

$$(2.12) \quad \limsup_{T \rightarrow \infty} \frac{1}{T} \int_a^T \int_a^t p(\tau) d\tau dt = \infty.$$

These theorems are true even for arbitrary $p(t)$ (not necessarily positive). In this case condition (2.12) is not sufficient for oscillation of the linear and superlinear cases. But as proved by Wong [42] and Hartman [26], (2.12) together with the following condition

$$\liminf_{t \rightarrow \infty} \int_a^t p(\tau) d\tau = -\lambda > -\infty, \quad \lambda > 0$$

guarantees the oscillation in both the linear and superlinear case. For more generalizations in this direction, we refer to Wong [45] and Butler and Erbe [11].

III. NONOSCILLATION CRITERIA FOR THE EQUATION

$$[r(x)y']' + p(x)y^\gamma = 0.$$

In the previous chapter we summarized the known oscillation and nonoscillation criteria for the equation

$$(3.1) \quad y'' + p(x)y^\gamma = 0, \gamma > 0.$$

In this chapter we shall prove some nonoscillation theorems for the more general equation

$$(3.2) \quad (r(x)y')' + p(x)y^\gamma = 0, \gamma > 0,$$

where $r(x)$ and $p(x)$ are positive functions and γ is the quotient of two positive odd integers. Thus (3.1) is the special case of (3.2) when $r(x) \equiv 1$. We extend theorems in Chapter II to much wider classes of $r(x)$, and most of our results can not be directly obtained via the changes of variables (2.3) and (2.4).

In section 1 we give some logarithm criteria which extend Theorem 2.13 of Erbe and Muldowney. For example we show that for a large class of functions $r(x)$ the condition that $r(x)\varphi(x)(\log x)^{\frac{\gamma-1}{\gamma}}$ is nonincreasing is sufficient for (3.2) to be nonoscillatory.

In section 2 we introduce the functions

$$Q_+(x) = \exp \int_a^x \frac{d(rp)_+}{rp} \quad \text{and} \quad Q_-(x) = \exp \int_a^x \frac{d(rp)_-}{rp}$$

and obtain a number of nonoscillation conditions in terms of $Q_+(x)$ and $Q_-(x)$.

These improve Erbe's results in [17].

In section 3 we consider the Jordan decompositions of the function $r(x)p(x)x^\delta$, where δ is a real number between 0 and $\frac{\gamma+3}{2}$, and obtain some conditions for (3.2) to possess oscillatory solutions.

§1 LOGARITHM CRITERIA FOR SUPERLINEAR EQUATIONS

In this section we consider the superlinear equation

$$(3.3) \quad (r(x)y'(x))' + p(x)y^\gamma(x) = 0, \gamma > 1.$$

The techniques we employ here are to make a standard change of variables and to introduce a suitable energy function. The possibility of doing so is pointed out in Erbe and Muldowney [20].

Now we make the change of variables

$$(3.4) \quad x = e^t, y = e^{\mu t} u, 0 < \mu \leq \frac{1}{2}.$$

Then we have that

$$\begin{aligned} \frac{dy}{dx} &= e^{(\mu-1)t} (\mu u + u') \\ \frac{d^2y}{dx^2} &= e^{(\mu-2)t} (\mu(\mu-1)u + (2\mu-1)u' + u''). \end{aligned}$$

Thus equation (3.3) is transformed into

$$(3.5) \quad (\rho(t)\epsilon(t)u')' + a(t, u)u = 0$$

where $\rho(t) = r(e^t)$, $\epsilon(t) = e^{(2\mu-1)t}$ and

$$a(t, u) = \epsilon(t)(\sigma(t)u^{\gamma-1} - \lambda(t))$$

$$\sigma(t) = p(e^t)e^{(\mu(\gamma-1)+2)t}$$

and

$$\lambda(t) = \mu[(1 - \mu)\rho(t) - \rho'(t)].$$

From the definition of $a(t, u)$, it is easily seen that when $\lambda(t) > 0$

$$(3.6) \quad a(t, u) > 0 \Leftrightarrow |u| > \alpha(t) = \left[\frac{\lambda(t)}{\sigma(t)} \right]^{\frac{1}{\gamma-1}}$$

and

$$(3.7) \quad \int_0^u a(t, s) s ds > 0 \Leftrightarrow |u| > \left[\frac{\gamma+1}{2} \right]^{\frac{1}{\gamma-1}} \alpha(t).$$

Now we prove a proposition which will be used in the proof of the main theorem of this section.

PROPOSITION 3.1. Suppose that (for some $0 < \mu \leq \frac{1}{2}$)

- (i) $\lambda(t)$ is positive and $\frac{d}{dt}[\rho(t)\lambda(t)\epsilon(t)] \geq 0$;
- (ii) $\frac{d}{dt}[\rho(t)\sigma(t)\epsilon(t)] \leq 0$;
- (iii) $\liminf_{t \rightarrow \infty} \int_t^{t+\delta} d[\epsilon^2(s)\rho(s)\lambda(s)\alpha^2(s)] > 0$ for some δ satisfying

$$0 < \frac{\delta}{\epsilon(\delta)} < \left(\frac{\gamma-1}{\mu(1-\mu)(\gamma-1)} \right)^{\frac{1}{2}} \left[\left(\frac{\gamma+1}{2} \right)^{\frac{1}{\gamma-1}} - 1 \right].$$

Then all nontrivial solutions of (3.5) are nonoscillatory.

Condition (i) implies that $\rho(t)$ is nondecreasing as we shall show in the following lemma.

LEMMA 3.2. If $\rho(t)[(1 - \mu)\rho(t) - \rho'(t)]$ is positive and nondecreasing, then $\rho(t)$ itself is nondecreasing.

Proof: Suppose $\frac{d}{dt}\rho(t_0) < 0$ for some $t_0 \in [a, \infty)$, then we can prove that $\rho(t)$ is decreasing on $[t_0, \infty)$. For assume t_1 is the first zero of $\rho'(t)$ exceeding t_0 . Then since $\rho(t)$ is positive and decreasing on $[t_0, t_1]$, we have

$$\begin{aligned} (1 - \mu)\rho^2(t_0) &> (1 - \mu)\rho^2(t_1) \\ &= (1 - \mu)\rho^2(t_1) - \rho(t_1)\rho'(t_1) \\ &\geq (1 - \mu)\rho^2(t_0) - \rho(t_0)\rho'(t_0) \\ &> (1 - \mu)\rho^2(t_0). \end{aligned}$$

This is a contradiction. Thus $\rho'(t)$ has no zeros on $[t_0, \infty)$, i.e.; $\frac{d}{dt}\rho(t) < 0$ there. Since $\rho(t)$ is positive and decreasing on $[t_0, \infty)$, $\rho^2(t)$ is also decreasing, thus by the condition that

$$(1 - \mu)\rho^2(t) - \rho(t)\rho'(t)$$

is nondecreasing, we have that $\rho(t)\rho'(t)$ is decreasing and negative. This implies that $-\rho'(t) = |\rho'(t)|$ is increasing. So $\rho(t)$ is concave down and this contradicts the fact that $\rho(t)$ is positive, therefore $\rho(t)$ is nondecreasing.

Note that since $0 < \mu \leq \frac{1}{2}$, the condition in the Lemma is implicitly contained in condition (i) of Proposition 3.1.

Proof of the proposition. Suppose $u(t)$ is an oscillatory solution of (3.5). Then we define the energy function

$$E(t) = (\rho(t)\epsilon(t)u'(t))^2 + 2\rho(t)\epsilon(t) \int_0^u a(t, s)ds.$$

By direct computation, we obtain

$$\begin{aligned}
 E'(t) &= (2\rho(t)\epsilon(t))' \int_0^u a(t,s) s ds + 2\rho(t)\epsilon(t) \int_0^u \frac{da(t,s)}{dt} s ds \\
 &= (2\rho(t)\epsilon(t))' \int_0^u \epsilon(t)(\sigma(t)s^{\gamma-1} - \lambda(t)) s ds + 2\rho(t)\epsilon^2(t) \int_0^u (\sigma'(t)s^{\gamma-1} - \lambda'(t)) s ds \\
 &= \epsilon(t) \left[\frac{2}{\gamma+1} (\rho(t)\epsilon(t)\sigma(t))' u^{\gamma+1} - (\rho(t)\epsilon(t)\lambda(t))' u^2 \right].
 \end{aligned}$$

It follows that $E'(t) \leq 0$ from conditions (i) and (ii). If $u(t) = 0$, then $u'(t) \neq 0$ since $u(t)$ is nontrivial. Thus $E(t) > 0$. Since $E(t)$ is nonincreasing and $u(t)$ has arbitrarily large zeros, $E(t)$ is positive on $[a, \infty)$. Thus when $u'(t) = 0$

$$E(t) = 2\rho(t)\epsilon(t) \int_0^u a(t,s) s ds > 0,$$

so by (3.7)

$$|u(t)| > \left[\frac{\gamma+1}{2} \right]^{\frac{1}{\gamma-1}} \alpha(t).$$

In particular, $|u(t)| > \alpha(t)$. We are going to show that this is true on an interval of fixed length ($> \delta$) containing t .

For fixed t , the minimal value of

$$2\rho(t)\epsilon(t) \int_0^u a(t,s) s ds$$

is achieved if $|u(t)| = \alpha(t)$ and is equal to

$$\begin{aligned}
 (3.8) \quad 2\rho(t)\epsilon(t) \int_0^{\pm\alpha(t)} a(t,s) s ds &= 2\rho(t)\epsilon(t) \int_0^{\pm\alpha(t)} \epsilon(t)(\sigma(t)s^{\gamma-1} - \lambda(t)) s ds, \\
 &= 2\rho(t)\epsilon^2(t) \left[\frac{\sigma(t)\alpha^{\gamma+1}(t)}{\gamma+1} - \frac{\lambda(t)\alpha^2(t)}{2} \right] \\
 &= 2\rho(t)\epsilon^2(t)\alpha^2(t) \left[\frac{\sigma(t) \cdot \frac{\lambda(t)}{\sigma(t)}}{\gamma+1} - \frac{\lambda(t)}{2} \right] \\
 &= -\frac{\gamma-1}{\gamma+1} \rho(t)\epsilon^2(t)\alpha^2(t)\lambda(t).
 \end{aligned}$$

Condition (i) and (ii) imply that $\alpha(t)$ is nondecreasing

$$\alpha'(t) = \frac{1}{\gamma-1} \left(\frac{\lambda(t)}{\sigma(t)} \right)^{\frac{2}{\gamma-1}} [(\rho(t)\epsilon(t)\gamma(t))'(\rho(t)\epsilon(t)\sigma(t)) - (\rho(t)\epsilon(t)\lambda(t))(\rho(t)\epsilon(t)\sigma(t))'] / (\rho(t)\epsilon(t)\sigma(t))^2 \geq 0.$$

Also condition (iii) implies that

$$\lim_{t \rightarrow \infty} \gamma(t)\epsilon^2(t)\lambda(t)\alpha^2(t) = \infty.$$

Now suppose t_n and t_{n+1} are successive zeros of $u(t)$ and $t_n < r_n < r_n < s_n < t_{n+1}$, where $|u(r_n)| = \alpha(r_n)$, $|u(s_n)| = \alpha(s_n)$ and $u'(r_n) = 0$. So

$$|u(r_n)| > \left[\frac{\gamma+1}{2} \right]^{\frac{1}{\gamma-1}} \alpha(r_n).$$

We assume that $u(t) > 0$ on (t_n, t_{n+1}) , the same argument may be applied to $-u(t)$ if $u(t) < 0$. Thus $u(t) > \alpha(t)$ on (r_n, s_n) . Since $a(t, u) > 0$ if $r_n < t < s_n$, from equation (3.5) we know $(\rho(t)\epsilon(t)u')' < 0$ on (r_n, s_n) . Therefore,

$$\begin{aligned} \int_{r_n}^{s_n} \rho(t)\epsilon(t)u'(t)dt &= \rho(\theta)\epsilon(\theta)u'(\theta)(r_n - r_n) \\ &\leq (r_n - r_n)\rho(r_n)\epsilon(r_n)u'(r_n). \end{aligned}$$

Thus

$$\begin{aligned} (r_n - r_n)u'(r_n) &\geq \int_{r_n}^{s_n} \frac{\rho(t)\epsilon(t)u'(t)}{\rho(r_n)\epsilon(r_n)} dt \\ &\geq \int_{r_n}^{s_n} \frac{\rho(t)\epsilon(r_n)}{\rho(r_n)\epsilon(r_n)} u'(t) dt. \end{aligned}$$

By Lemma 3.2, $\rho(t)$ is nondecreasing on $[a, \infty)$. Since $\alpha(t)$ is nondecreasing and $\alpha(r_n) = u(r_n)$, we have that

$$\begin{aligned} (r_n - r_n)u'(r_n) &\geq \int_{r_n}^{r_n} \frac{\epsilon(r_n)}{\epsilon(r_n)} u'(t) dt \\ &= \frac{\epsilon(r_n)}{\epsilon(r_n)} [u(r_n) - u(r_n)] \\ &\geq \frac{\epsilon(r_n)}{\epsilon(r_n)} \left[\left(\frac{\gamma+1}{2} \right)^{\frac{1}{\gamma-1}} \alpha(r_n) - \alpha(r_n) \right] \\ &\geq \frac{\epsilon(r_n)}{\epsilon(r_n)} \alpha(r_n) \left[\left(\frac{\gamma+1}{2} \right)^{\frac{1}{\gamma-1}} - 1 \right]. \end{aligned}$$

That is,

$$(r_n - r_n) \frac{\epsilon(r_n)}{\epsilon(r_n)} u'(r_n) \geq \alpha(r_n) \left[\left(\frac{\gamma+1}{2} \right)^{\frac{1}{\gamma-1}} - 1 \right].$$

Because $E(t)$ is decreasing, $E(r_n) \leq E(a)$ and by (3.8),

$$\begin{aligned} E(r_n) &= [\rho(r_n)\epsilon(r_n)u'(r_n)]^2 + 2\rho(r_n)\epsilon(r_n) \int_0^{\alpha(r_n)} a(r_n, s) ds \\ &= [\rho(r_n)\epsilon(r_n)u'(r_n)]^2 - \frac{\gamma-1}{\gamma+1} \rho(r_n)\epsilon^2(r_n)\alpha^2(r_n)\lambda(r_n), \end{aligned}$$

i.e., $\rho^2(r_n)\epsilon^2(r_n)(u'(r_n))^2 \leq E(a) + \frac{\gamma-1}{\gamma+1} \rho(r_n)\epsilon^2(r_n)\alpha^2(r_n)\lambda(r_n)$. This implies that

$$u'(r_n) \leq \left[\frac{E(a) + \frac{\gamma-1}{\gamma+1} \rho(r_n)\epsilon^2(r_n)\alpha^2(r_n)\lambda(r_n)}{\rho^2(r_n)\epsilon^2(r_n)} \right]^{\frac{1}{2}}.$$

Suppose c_1 is a constant greater than $\sqrt{\frac{\gamma-1}{\gamma+1}}$, then if r_n is large enough,

$$\begin{aligned} u'(r_n) &\leq c_1 \alpha(r_n) \left[\frac{\lambda(r_n)}{\rho(r_n)} \right]^{\frac{1}{2}} \\ &= c_1 \alpha(r_n) \left[\frac{\mu((1-\mu)\rho(r_n) - \rho'(r_n))}{\rho(r_n)} \right]^{\frac{1}{2}}. \end{aligned}$$

Since $\rho(t)$ is nondecreasing, $\frac{\rho'(t)}{\rho(t)} \geq 0$. Therefore

$$u'(r_n) \leq c_1 \alpha(r_n) [\mu(1-\mu)]^{\frac{1}{\gamma+1}} = c \alpha(r_n),$$

where $c = c_1(\mu(1-\mu))^{\frac{1}{\gamma+1}} > 0$. Finally we obtain

$$\begin{aligned} \alpha(r_n) \left[\left(\frac{\gamma+1}{2} \right)^{\frac{1}{\gamma+1}} - 1 \right] &\leq (r_n - r_n) \frac{\epsilon(r_n)}{\epsilon(r_n)} u'(r_n) \\ &\leq c(r_n - r_n) \frac{\epsilon(r_n)}{\epsilon(r_n)} \alpha(r_n). \end{aligned}$$

This means

$$(r_n - r_n) \frac{\epsilon(r_n)}{\epsilon(r_n)} \geq c^{-1} \left[\left(\frac{\gamma+1}{2} \right)^{\frac{1}{\gamma+1}} - 1 \right].$$

Since

$$\delta \cdot \epsilon^{-1}(\delta) < c^{-1} \left[\left(\frac{\gamma+1}{2} \right)^{\frac{1}{\gamma+1}} - 1 \right],$$

we obtain the result we claimed, i.e., that

$$s_n - r_n > r_n - r_n > \delta.$$

Now we estimate $E'(t)$ on (r_n, s_n) . Since $|u(t)| > \alpha(t)$ on (r_n, s_n) ,

$$\begin{aligned} E'(t) &= \epsilon(t) \left[\frac{2}{\gamma+1} (\rho(t)\epsilon(t)\sigma(t))' u^{\gamma+1}(t) - (\rho(t)\lambda(t)\epsilon(t))' u^2(t) \right] \\ &\leq \epsilon(t) \left[\frac{2}{\gamma+1} (\rho(t)\sigma(t)\epsilon(t))' \alpha^{\gamma+1}(t) - (\rho(t)\lambda(t)\epsilon(t))' \alpha^2(t) \right] \\ &= \epsilon(t) \left[\frac{2}{\gamma+1} (\rho(t)\sigma(t)\epsilon(t))' \left(\frac{\lambda(t)}{\sigma(t)} \right)^{\frac{\gamma+1}{\gamma+1}} - (\rho(t)\lambda(t)\epsilon(t))' \left(\frac{\lambda(t)}{\sigma(t)} \right)^{\frac{2}{\gamma+1}} \right] \\ &= \epsilon(t) \left[\frac{2}{\gamma+1} (\rho(t)\sigma(t)\epsilon(t))' \left(\frac{\rho(t)\lambda(t)\epsilon(t)}{\rho(t)\sigma(t)\epsilon(t)} \right)^{\frac{\gamma+1}{\gamma+1}} - (\rho(t)\lambda(t)\epsilon(t))' \left(\frac{\rho(t)\lambda(t)\epsilon(t)}{\rho(t)\sigma(t)\epsilon(t)} \right)^{\frac{2}{\gamma+1}} \right] \\ &= \epsilon(t) \left[-\frac{\gamma-1}{\gamma+1} \cdot \frac{(\rho(t)\lambda(t)\epsilon(t))^{\frac{\gamma+1}{\gamma+1}}}{\rho(t)\sigma(t)\epsilon(t)^{\frac{2}{\gamma+1}}} \right]' \\ &= -\frac{\gamma-1}{\gamma+1} \epsilon(t) [\rho(t)\lambda(t)\alpha^2(t)\epsilon(t)]'. \end{aligned}$$

i.e.,

$$dE(t) \leq -\frac{\gamma-1}{\gamma+1} \epsilon(t) d[\rho(t)\lambda(t)\alpha^2(t)\epsilon(t)], \text{ if } r_n \leq t \leq s_n.$$

Therefore by condition (iii) and integration by parts we obtain

$$\begin{aligned} E(\infty) - E(t_0) &= \int_{t_0}^{\infty} dE(t) \\ &\leq \sum_n \int_{r_n}^{s_n} dE(t) \\ &\leq -\frac{\gamma-1}{\gamma+1} \sum_n \int_{r_n}^{s_n} \epsilon(t) d[\rho(t)\lambda(t)\alpha^2(t)\epsilon(t)] \\ &= -\frac{\gamma-1}{\gamma+1} \sum_n \left[\int_{r_n}^{s_n} d(\rho(t)\lambda(t)\alpha^2(t)\epsilon(t)) + \right. \\ &\quad \left. \int_{r_n}^{s_n} (1-2\mu)\rho(t)\lambda(t)\alpha^2(t)\epsilon^2(t) dt \right] \\ &\leq -\frac{\gamma-1}{\gamma+1} \sum_n \int_{r_n}^{s_n} d[\rho(t)\lambda(t)\alpha^2(t)\epsilon^2(t)] = -\infty. \end{aligned}$$

Since $E(t) \geq 0$, this is a contradiction. So we know that equation (3.5) has no oscillatory solutions.

Now we are ready to prove our main theorem of this section.

THEOREM 3.3. Suppose that

(i) $r(x)[(1-\mu)r(x) - xr'(x)]x^{2\mu-1}$ is positive and nondecreasing;

(ii) $(r(x)p(x)x^{\mu(\gamma+1)+1})' \leq 0$;

(iii) $\liminf_{s \rightarrow \infty} \int_s^{ks} d[x^{2\mu-1}(r(x)p(x)x^{\mu(\gamma+1)+1})]^{\frac{\gamma}{\gamma-1}} > 0$, where $k = e^\delta$

then all solutions of (3.3) are nonoscillatory.

Proof: Conditions (i) and (ii) here are the same as those of the proposition. We verify condition (iii).

$$\begin{aligned}\epsilon^2(t)(\rho(t)\lambda(t)\alpha^2(t)) &= [\epsilon(t)\rho(t)\lambda(t)]\alpha^2(t)\epsilon(t) \\ &= [\epsilon(t)\rho(t)\lambda(t)] \cdot \epsilon(t) \left(\frac{\lambda(t)\rho(t)\epsilon(t)}{\sigma(t)\rho(t)\epsilon(t)} \right)^{\frac{2}{\gamma-1}} \\ &= (\epsilon(t)\rho(t)\lambda(t))^{\frac{\gamma+1}{\gamma-1}} \epsilon(t)(\sigma(t)\gamma(t)\epsilon(t))^{\frac{-2}{\gamma-1}}.\end{aligned}$$

Since $(\epsilon(t)\rho(t)\lambda(t))^{\frac{\gamma+1}{\gamma-1}} = [r(x)((1-\mu)r(x) - xr'(x))x^{2\mu-1}]^{\frac{\gamma+1}{\gamma-1}}$ is nondecreasing condition (iii) here implies condition (iii) of the proposition.

COROLLARY 3.4. Suppose that

(i) $r(x)[\frac{1}{2}r(x) - xr'(x)]$ is positive and nondecreasing;

(ii) $(r(x)p(x)x^{\frac{\gamma+1}{2}})' \leq 0$;

(iii) $\liminf_{s \rightarrow \infty} \int_s^{ks} d[(r(x)p(x)x^{\frac{\gamma+1}{2}})^{\frac{-2}{\gamma-1}}] > 0$;

then all solutions of (3.3) are nonoscillatory.

This is just the special case of Theorem 3.3 when $\mu = \frac{1}{2}$.

COROLLARY 3.5. Suppose that for some $0 < \mu \leq \frac{1}{2}$,

(i) $r(x)[(1-\mu)r(x) - xr'(x)]x^{2\mu-1}$ is positive and nondecreasing;

(ii) $r(x)p(x)x^{2\mu+\frac{\gamma+1}{2}}(\log x)^{\frac{\gamma-1}{2}}$ is nonincreasing;

then all solutions of (3.3) are nonoscillatory.

Proof: Notice that

$$\begin{aligned}r(x)p(x)x^{2\mu+\frac{\gamma+1}{2}}(\log x)^{\frac{\gamma-1}{2}} &= (r(x)p(x)x^{\mu(z+1)+1})x^{\frac{\gamma-1}{2}(1-2\mu)}(\log x)^{\frac{\gamma-1}{2}} \\ &= (r(x)p(x)x^{\mu(z+1)+1})(x^{1-2\mu}\log x)^{\frac{\gamma-1}{2}}.\end{aligned}$$

Since $r(x)p(x)x^{\mu(\gamma+1)+1} (x^{1-2\mu} \log x)^{\frac{\gamma-1}{2}}$ is nonincreasing and $\mu \leq \frac{1}{2}$, $r(x)p(x)x^{\mu(\gamma+1)+1}$ is also nonincreasing, this is condition (ii) of the Theorem 3.3.

Also we see that

$$\begin{aligned} & \int_x^{kx} d[x^{2\mu-1} (r(x)p(x)x^{\mu(\gamma+1)})^{\frac{\gamma-1}{2}}] \\ &= \int_x^{kx} d[\log x (r(x)p(x)x^{2\mu+\frac{\gamma+1}{2}} (\log x)^{\frac{\gamma-1}{2}})^{\frac{\gamma-1}{2}}]. \end{aligned}$$

Since $[r(x)p(x)x^{2\mu+\frac{\gamma+1}{2}} (\log x)^{\frac{\gamma-1}{2}}]^{\frac{\gamma-1}{2}}$ is nondecreasing, thus there exists a constant $c > 0$ independent of x , such that

$$\begin{aligned} & \int_x^{kx} d[\log x (r(x)p(x)x^{2\mu+\frac{\gamma+1}{2}} (\log x)^{\frac{\gamma-1}{2}})^{\frac{\gamma-1}{2}}] \\ & \geq C \cdot \int_x^{kx} d(\log x) = ck > 0. \end{aligned}$$

By the theorem, all solutions of (3.3) are nonoscillatory.

COROLLARY 3.6. Suppose that

- (i) $r(x)[\frac{1}{2}r(x) - xr'(x)]$ is positive and nondecreasing;
- (ii) $r(x)p(x)x^{\frac{\gamma+3}{2}} (\log x)^{\frac{\gamma-1}{2}}$ is nonincreasing;

then all solutions of (3.3) are nonoscillatory.

This is the case of $\mu = \frac{1}{2}$ in Corollary 3.5. When $r(x) \equiv 1$, our corollary gives a criterion similar to Chiou's and Erbe and Muldowney's logarithm criterion. We now give some applications to more general equations.

Example: Let k_1, k_2 be two real numbers satisfying

$$0 \leq k_1 < 1 \text{ and } k_2 < \min\{-k_1, k_1 - 1\} - \frac{\gamma+1}{2}.$$

Then the equation

$$(x^{k_1} y')' + x^{k_2} y^\gamma = 0, \gamma > 1$$

is nonoscillatory. For if $k_1 < \frac{1}{2}$, then we apply Corollary 3.5 with $\mu = \frac{1}{2} - k_1$, we have

$$\begin{aligned} r(x)[(1-\mu)r(x) - xr'(x)]x^{2\mu-1} &= x^{k_1}[(1-\mu)x^{k_1} - k_1 x^{k_1}]x^{2\mu-1} \\ &= (1-\mu-k_1)x^{2k_1+2\mu-1} = \frac{1}{2} \end{aligned}$$

and

$$r(x)p(x)x^{2\mu+\frac{\gamma+1}{2}}(\log x)^{\frac{\gamma-1}{2}} = x^{k_1+k_2+2\mu+\frac{\gamma+1}{2}}(\log x)^{\frac{\gamma-1}{2}}.$$

Since $1 - k_1 + k_2 + \frac{\gamma+1}{2} < 0$, conditions (i) and (ii) of Corollary 3.5 are satisfied.

When $k_1 \geq \frac{1}{2}$, we choose μ such that

$$0 < \mu < \min\left\{1 - k_1, -\frac{k_1 + k_2}{2} - \frac{1 + \gamma}{4}\right\}.$$

then since $\mu < 1 - k_1$ and $2k_1 + 2\mu - 1 > 2k_1 - 1 \geq 0$, $r(x)[(1-\mu)r(x) - xr'(x)]x^{2\mu-1}$ is positive and nondecreasing. Also since $k_1 + k_2 + 2\mu + \frac{\gamma+1}{2} < 0$, $r(x)p(x)x^{2\mu+\frac{\gamma+1}{2}}(\log x)^{\frac{\gamma-1}{2}}$ is nonincreasing, so again by Corollary 3.5 the equation is nonoscillatory.

§2 SOME OTHER NONOSCILLATION CRITERIA

This section is for both superlinear and sublinear cases. The equation under consideration is

$$(3.9) \quad (r(x)y'(x))' + p(x)y^\gamma(x) = 0, \gamma > 0.$$

We suppose that the product function $r(x)p(x)$ is locally of absolutely continuous in $[a, +\infty)$. Let

$$rp(x) = rp(a) + rp_+(x) - rp_-(x)$$

be the Jordan decomposition of rp , rp_- and rp_+ are nondecreasing and $rp_+(a) = rp_-(a) = 0$. Similar to Erbe [17] we define the functions

$$Q_+(x) = \exp \int_a^x \frac{d(rp_+(t))}{rp(t)}; \quad Q_-(x) = \exp \int_a^x \frac{d(rp_-(t))}{rp(t)}.$$

Since $d(rp_+) \geq 0 \geq -d(rp_-)$, both $Q_+(x)$ and $Q_-(x)$ are increasing. Also

$$\begin{aligned} \frac{Q_+(x)}{Q_-(x)} &= \exp \left(\int_a^x \frac{d(rp_+(t))}{rp(t)} - \int_a^x \frac{d(rp_-(t))}{rp(t)} \right) \\ &= \exp \int_a^x \frac{d(rp(t))}{rp(t)} = \exp(\ln rp(x) - \ln rp(a)) \\ &= \frac{rp(x)}{rp(a)}. \end{aligned}$$

We define for a solution $y(x)$ of (3.9) two more functions

$$E(x) = \frac{(r(x)y'(x))^2}{2} + \frac{rp(x)y^{\gamma+1}(x)}{\gamma+1}$$

and

$$B(x) = \frac{E(x)}{rp(x)}.$$

LEMMA 3.7. For $x \geq \xi \geq a$, we have

$$\frac{Q_+(\xi)}{Q_+(x)} \leq \frac{E(\xi)}{E(x)} \leq \frac{Q_-(x)}{Q_-(\xi)}$$

and

$$\frac{Q_-(\xi)}{Q_-(x)} \leq \frac{B(\xi)}{B(x)} \leq \frac{Q_+(x)}{Q_+(\xi)}.$$

Proof: We calculate that

$$\begin{aligned}
 E'(x) &= (r(x)y'(x))(r(x)y'(x))' + rp(x)y^\gamma y'(x) + \frac{y^{\gamma+1}(x)}{\gamma+1}(rp(x))' \\
 &= \frac{y^{\gamma+1}(x)}{\gamma+1}(rp)'(x) \\
 &= \frac{y^{\gamma+1}(x)}{\gamma+1}[(rp_+)'(x) - (rp_-)'(x)].
 \end{aligned}$$

Thus

$$\begin{aligned}
 \frac{(rp_+)'(x)}{rp(x)} \cdot E(x) &= \frac{(rp_+)'(x)}{rp(x)} \left[\frac{(r(x)y'(x))^2}{2} + \frac{rp(x)y^{\gamma+1}(x)}{\gamma+1} \right] \\
 &\geq \frac{(rp_+)'(x)}{rp(x)} \cdot \frac{rp(x)y^{\gamma+1}(x)}{\gamma+1} \\
 &= \frac{(rp_+)'(x)y^{\gamma+1}}{\gamma+1} \geq E'(x) \\
 &\geq -\frac{y^{\gamma+1}(x)}{\gamma+1}(rp_-)'(x) \\
 &= -\frac{(rp_-)'(x)}{rp(x)}(rp(x)\frac{y^{\gamma+1}(x)}{\gamma+1}) \\
 &\geq -\frac{(rp_-)'(x)}{rp(x)}E(x),
 \end{aligned}$$

$$\text{i.e., } \frac{(rp_+)'(x)}{rp(x)}E(x) \geq E'(x) \geq -\frac{(rp_-)'(x)}{rp(x)}E(x) \text{ or } \frac{(rp_+)'(x)}{rp(x)} \geq \frac{E'(x)}{E(x)} \geq -\frac{(rp_-)'(x)}{rp(x)}.$$

Integrate from ξ to x , we obtain

$$\frac{Q_+(x)}{Q_+(\xi)} = \exp \int_{\xi}^x \frac{d(rp_+)(t)}{rp(t)} \geq \frac{E(x)}{E(\xi)} \geq \exp(-\int_{\xi}^x \frac{d(rp_-)(t)}{rp(t)}) = \frac{Q_-(\xi)}{Q_-(x)}.$$

Similarly,

$$\begin{aligned}
 B'(x) &= \frac{E'(x)(rp(x)) - E(x)(rp)'(x)}{(rp)^2(x)} \\
 &= \frac{1}{rp(x)} \left[\frac{y^{\gamma+1}(x)}{\gamma+1} ((rp_+)'(x) - (rp_-)'(x)) - \frac{E(x)}{rp(x)} ((rp_+)'(x) - (rp_-)'(x)) \right] \\
 &= [(rp_+)'(x) - (rp_-)'(x)] \frac{1}{rp(x)} \left(\frac{y^{\gamma+1}(x)}{\gamma+1} - \frac{E(x)}{rp(x)} \right) \\
 &= -[(rp_+)'(x) - (rp_-)'(x)] \frac{(r(x)y'(x))^2}{2(rp)^2(x)} \\
 &= -\frac{1}{2} \left(\frac{y'}{p} \right)^2 [(rp_+)'(x) - (rp_-)'(x)]
 \end{aligned}$$

$$\begin{aligned}
\frac{-(rp_+)'(x)}{rp(x)} B(x) &= -\frac{(rp_+)'(x)}{(rp)(x)} \cdot \frac{E(x)}{rp(x)} \\
&= \frac{-(rp_+)'(x)}{(rp)^2(x)} \left[\frac{(r(x)y(x))^2}{2} + \frac{rp(x)y^{\gamma+1}(x)}{\gamma+1} \right] \\
&\leq \frac{-(rp_+)'(x)}{2(rp)^2(x)} \leq B'(x) \leq \frac{(rp_-)'(x)}{(rp)^2(x)} \cdot \frac{(r(x)y(x))^2}{2} \\
&\leq \frac{(rp_-)'(x)}{(rp)^2(x)} \cdot E(x) = \frac{(rp_-)'(x)}{rp(x)} \cdot B(x),
\end{aligned}$$

i.e., $-\frac{(rp_+)'(x)}{rp(x)} \leq \frac{B'(x)}{B(x)} \leq \frac{(rp_-)'(x)}{rp(x)}$. Integrate again from ξ to x ,

$$\frac{Q_+(\xi)}{Q_+(x)} = \exp\left(-\int_{\xi}^x \frac{d(rp_+)(t)}{rp(t)}\right) \leq \frac{B(x)}{B(\xi)} \leq \exp\left(\int_{\xi}^x \frac{d(rp_-)(t)}{rp(t)}\right) \leq \frac{Q_-(x)}{Q_-(\xi)},$$

i.e., $\frac{Q_+(x)}{Q_+(\xi)} \geq \frac{B(\xi)}{B(x)} \geq \frac{Q_-(\xi)}{Q_-(x)}$.

LEMMA 3.8. Suppose $r(x) > 0$ and satisfies $\int_a^{\infty} \frac{dx}{r(x)} = \infty$, and $\limsup_{x \rightarrow \infty} \int_a^x \frac{dt}{r(t)} \cdot \int_x^{\infty} p(t) dt < \frac{1}{4}$, then the linear equation

$$(r(x)y'(x))' + p(x)y(x) = 0$$

is nonoscillatory.

Proof: We make the substitution $s(x) = \int_a^x \frac{dt}{r(t)}$, then the above equation becomes

$$\frac{d^2 y}{ds^2} + r(x(s))p(x(s))y = 0,$$

and

$$\begin{aligned}
S \int_s^{\infty} r(x(s))p(x(s))ds &= s \int_{s(s)}^{\infty} r(x)p(x) \frac{dx}{r(x)} \\
&= \int_a^s \frac{dt}{r(t)} \int_s^{\infty} p(t) dt.
\end{aligned}$$

we have the condition

$$\limsup_{z \rightarrow \infty} \int_a^z \frac{dt}{r(t)} \cdot \int_z^\infty p(t) dt < \frac{1}{4},$$

which means

$$\lim_{s \rightarrow \infty} S \cdot \int_S^\infty r(x(s))p(x(s)) ds < \frac{1}{4}.$$

By Hille's criterion (Theorem 2.1), the equation $(r(x)y'(x))' + p(x)y(x) = 0$ is nonoscillatory.

THEOREM 3.9. *Let $\gamma \geq 1$. Then equation (3.9) is nonoscillatory in case any of the following hold:*

- (a) $\int_a^\infty (\int_a^t \frac{dr}{r})^{\gamma-1} p(t)(Q_+(t))^{\frac{\gamma-1}{\gamma}} dt < \infty$, $\int_a^\infty \frac{dt}{r(t)} = \infty$ and $\lim_{z \rightarrow \infty} \int_a^z \frac{dt}{r(t)} \int_z^\infty (\int_a^t \frac{dr}{r})^{\gamma-1} p(t)(Q_+(t))^{\frac{\gamma-1}{\gamma}} dt = 0$;
- (b) $\int_a^\infty (p(t))^{\frac{\gamma}{\gamma+1}} (Q_+(t)/r(t))^{\frac{1}{\gamma+1}} dt < \infty$, $\int_a^\infty \frac{dt}{r(t)} = \infty$ and $\lim_{z \rightarrow \infty} \int_a^z \frac{dt}{r(t)} \int_z^\infty (p(t))^{\frac{\gamma}{\gamma+1}} (Q_+(t)/r(t))^{\frac{1}{\gamma+1}} dt = 0$;
- (c) $\int_a^\infty p(t)(Q_-(t))^{\frac{\gamma}{\gamma+1}} dt < \infty$, $\int_a^\infty \frac{dt}{r(t)} = \infty$ and $\lim_{z \rightarrow \infty} \int_a^z \frac{dt}{r(t)} \int_z^\infty p(t)(Q_-(t))^{\frac{\gamma}{\gamma+1}} dt = 0$;
- (d) $\int_a^\infty (\int_a^t \frac{dr}{r})^\gamma p(t) dt < \infty$, and $\lim_{z \rightarrow \infty} (Q_+(z))^{\frac{\gamma-1}{\gamma}} \int_z^\infty (\int_a^t \frac{dr}{r})^\gamma p(t) dt = 0$;
- (e) $\int_a^\infty (p(t))^{\frac{1}{\gamma+1}} (Q_+(t)/r(t))^{\frac{1}{\gamma+1}} dt < \infty$;
- (f) $\int_a^\infty \frac{dt}{r(t)} = \infty$ and $\lim_{z \rightarrow \infty} (\int_a^z \frac{dt}{r(t)})^{\gamma+1} p(z)r(z)(Q_+(z))^{\frac{\gamma-1}{\gamma}} = 0$;

$$(g) \int_a^\infty \frac{dt}{r(t)} = \infty \text{ and}$$

$$\lim_{x \rightarrow \infty} \left(\int_a^x \frac{dt}{r(t)} \right)^2 p(x) r(x) (Q_-(x))^{\frac{\gamma-1}{\gamma}} = 0.$$

Proof: (a) Any solution $y(x)$ of equation (3.9) is also a solution of the linear equation

$$(r(x)z'(x))' + p(x)(y(x))^{\gamma-1} z(x) = 0.$$

From Lemma 3.7, we have

$$(r(x)y'(x))^2 \leq 2E(x) \leq 2E(a)Q_+(x), \text{ here } \xi = a.$$

This implies

$$|r(x)y'(x)| \leq (2E(a))^{\frac{1}{2}} (Q_+(x))^{\frac{1}{2}}$$

or

$$|y'(x)| \leq (2E(a))^{\frac{1}{2}} \frac{(Q_+(x))^{\frac{1}{2}}}{r(x)}.$$

By an integration, we get

$$\begin{aligned} |y(x)| &\leq k_0 + (2E(a))^{\frac{1}{2}} \int_a^x \frac{(Q_+(t))^{\frac{1}{2}}}{r(t)} dt \\ &\leq k_0 + k_1 (Q_+(x))^{\frac{1}{2}} \int_a^x \frac{dt}{r(t)}, \end{aligned}$$

where k_0, k_1 are positive constants. Hence for large x we have

$$|y(x)| \leq k(Q_+(x))^{\frac{1}{2}} \int_a^x \frac{dt}{r(t)}.$$

Thus

$$\begin{aligned} \limsup_{x \rightarrow \infty} \int_a^x \frac{dt}{r(t)} \int_x^\infty p(t)(y(t))^{\gamma-1} dt \\ \leq \limsup_{x \rightarrow \infty} k^{\gamma-1} \int_a^x \frac{dt}{r(t)} \cdot \int_x^\infty p(t)[Q_+(x)]^{\frac{\gamma-1}{2}} \left(\int_a^t \frac{dr}{r(r)} \right)^{\gamma-1} dt = 0. \end{aligned}$$

So by Lemma 3.8, $y(x)$ is nonoscillatory.

(c) The proof is similar to that of part (a). By Lemma 3.7 and from

$$\frac{y^{\gamma+1}(x)r(x)p(x)}{\gamma+1} \leq E(x)$$

We have

$$\frac{y^{\gamma+1}(x)}{\gamma+1} \leq \frac{E(x)}{rp(x)} = B(x),$$

we obtain

$$|y(x)| \leq [(\gamma+1)B(a)(Q_-(x))]^{\frac{1}{\gamma+1}}.$$

Therefore

$$\begin{aligned} \limsup_{x \rightarrow \infty} \int_a^x \frac{dt}{r(t)} \int_x^\infty p(t)|y(t)|^{\gamma-1} dt \\ \leq \limsup_{x \rightarrow \infty} c \int_a^x \frac{dt}{r(t)} \int_x^\infty p(t)[Q_-(t)]^{\frac{\gamma-1}{\gamma+1}} dt = 0. \end{aligned}$$

By Lemma 3.8, $y(x)$ is nonoscillatory.

(b) This can be proved by part (c) and the relation

$$(*) \quad \frac{Q_+(x)}{Q_-(x)} = \frac{rp(x)}{rp(a)}$$

(d) Suppose that $y(x)$ is an oscillatory solution of (3.9) and let $x_n \rightarrow \infty$ be a sequence such that $y(x_n) = 0$ and $y'(x_n) > 0$. Let $z_n > x_n$ be the first zero of $y'(x)$ exceeding x_n . Then for $x_n < x < z_n$, we have $y'(x) > 0$ and $(r(x)y'(x))' < 0$.

So from equation (3.9), we get

$$\begin{aligned}
 r(x_n)y'(x_n) &= \int_{x_n}^{x_n} p(t)(y(t))^\gamma dt \\
 &= \int_{x_n}^{x_n} p(t) \left(\int_{x_n}^t y'(s) ds \right)^\gamma dt \\
 &= \int_{x_n}^{x_n} p(t) \left(\int_{x_n}^t \frac{r(s)y'(s)}{r(s)} ds \right)^\gamma dt \\
 &\leq (r(x_n)y'(x_n))^\gamma \int_{x_n}^{x_n} p(t) \left(\int_{x_n}^t \frac{ds}{r(s)} \right)^\gamma dt.
 \end{aligned}$$

By Lemma 3.7, $|r(x_n)y'(x_n)| \leq C \cdot (Q_+(x_n))^{\frac{1}{2}}$, where C is a constant, so

$$\begin{aligned}
 1 &\leq (r(x_n)y'(x_n))^{\gamma-1} \int_{x_n}^{\infty} p(t) \left(\int_{x_n}^t \frac{ds}{r(s)} \right)^\gamma dt \\
 &\leq C^{\gamma-1} (Q_+(x_n))^{\frac{\gamma-1}{2}} \int_{x_n}^{\infty} p(t) \left(\int_{x_n}^t \frac{ds}{r(s)} \right)^\gamma dt.
 \end{aligned}$$

This is a contradiction to

$$\lim_{x \rightarrow \infty} (Q_+(x))^{\frac{\gamma-1}{2}} \int_x^{\infty} \left(\int_x^t \frac{d\tau}{r(\tau)} \right)^\gamma p(t) dt = 0.$$

(e) Suppose that $y(x)$ is an oscillatory solution of (3.9). Let $x_n \rightarrow \infty$ such that $y(x_n) = 0$ and $y'(x_n) > 0$. Let $x_n \rightarrow \infty$, $y'(x_n) = 0$, $x_n > x_n$ be the first zero exceeding x_n and put $\xi = x_n$ in Lemma 3.7. Then $x > x_n$ implies that

$$E(x) \leq E(x_n) \frac{Q_+(x)}{Q_+(x_n)}.$$

Since $E(x_n) = \frac{(ry')^2(x)}{2}$,

$$\frac{rp(x)y^{\gamma+1}(x)}{\gamma+1} \leq \frac{(ry')^2(x_n)}{2} \cdot \frac{Q_+(x)}{Q_+(x_n)},$$

which implies

$$|y(x)|^\gamma \leq C \cdot (r(x_n)y'(x_n))^{\frac{2}{\gamma+1}} (Q_+(x))^{\frac{1}{\gamma+1}} (Q_+(x_n))^{\frac{1}{\gamma+1}} (r(x)p(x))^{\frac{1}{\gamma+1}}.$$

Therefore,

$$\begin{aligned} - \int_{x_n}^{x_n} (r(t)(y'(t))') dt &= r(x_n)y'(x_n) = \int_{x_n}^{x_n} p(t)y^\gamma(t) dt \\ &\leq C(r(x_n)y'(x_n))^{\frac{2}{\gamma+1}} (Q_+(x_n))^{\frac{1}{\gamma+1}} \int_{x_n}^{x_n} (r(t))^{\frac{1}{\gamma+1}} \\ &\quad \int_{x_n}^{x_n} (r(t))^{\frac{1}{\gamma+1}} (p(t))^{\frac{1}{\gamma+1}} (Q_+(t))^{\frac{1}{\gamma+1}} dt. \end{aligned}$$

Dividing by $(r(x_n)y'(x_n))^{\frac{2}{\gamma+1}}$, we have

$$\begin{aligned} (r(x_n)y'(x_n))^{\frac{1}{\gamma+1}} &\leq C(Q_+(x_n))^{\frac{1}{\gamma+1}} \int_{x_n}^{\infty} (r(t))^{\frac{1}{\gamma+1}} (p(t))^{\frac{1}{\gamma+1}} (Q_+(t))^{\frac{1}{\gamma+1}} dt \\ (3.10) \quad &= C(Q_+(x_n))^{\frac{1}{\gamma+1}} \int_{x_n}^{\infty} (p(t))^{\frac{1}{\gamma+1}} \left(\frac{Q_+(t)}{r(t)}\right)^{\frac{1}{\gamma+1}} dt. \end{aligned}$$

From Lemma 3.7 with $\xi = a$, one can get

$$E(x_n) \leq \frac{E(a)}{Q_+(a)} Q_+(x_n) \equiv C Q_+(x_n).$$

So

$$(r(x_n)y'(x_n))^2 \leq C_1 Q_+(x_n).$$

Since $\gamma \geq 1$

$$(3.11) \quad (r(x_n)y'(x_n))^{\frac{1}{\gamma+1}} \geq C_2 [Q_+(x_n)]^{\frac{1}{\gamma+1}}.$$

From (3.10) and (3.11),

$$C_3 (Q_+(x_n))^{\frac{1}{2}} \leq \int_{x_n}^{\infty} (p(t))^{\frac{1}{\gamma+1}} \left(\frac{Q_+(t)}{r(t)}\right)^{\frac{1}{\gamma+1}} dt.$$

Since the right hand side tends to zero and the left hand side is nondecreasing, we obtain a contradiction.

(f) With the assumption $\int_a^\infty \frac{dt}{r(t)} = \infty$, we have for $\xi > a$

$$\int_\xi^z \left(\int_a^t \frac{d\tau}{r(\tau)} \right)^{-2} \frac{dt}{r(t)} = \left(\int_a^\xi \frac{d\tau}{r(\tau)} \right)^{-1} - \left(\int_a^z \frac{d\tau}{r(\tau)} \right)^{-1}.$$

Thus

$$\int_\xi^\infty \left(\int_a^t \frac{1}{r} \right)^{-2} \frac{dt}{r(t)} < \infty.$$

This and the condition in part (f) imply that

$$\int_a^\infty \left(\int_a^t \frac{d\tau}{r(\tau)} \right)^{\gamma-1} p(t)(Q_+(t))^{\frac{\gamma-1}{2}} dt < \infty.$$

An application of L'Hospital's rule shows that

$$\lim_{z \rightarrow \infty} \int_a^z \frac{dt}{r(t)} \int_z^\infty \left(\int_a^t \frac{d\tau}{r(\tau)} \right)^{\gamma-1} p(t)(Q_+(t))^{\frac{\gamma-1}{2}} dt = 0.$$

This is just the condition in part (a).

(g) This can be proved in the same way as part (f), but we apply part (c) instead of part (a).

COROLLARY 3.10. *The linear equation*

$$(r(x)y'(x))' + p(x)y(x) = 0$$

is nonoscillatory in case any of the following hold:

$$(a) \int_a^\infty \left(\int_a^t \frac{d\tau}{r(\tau)} \right) p(t) dt < \infty;$$

$$(b) \int_a^\infty \left(\frac{p(t)Q_+(t)}{r(t)} \right)^{\frac{1}{2}} dt < \infty;$$

$$(c) \int_a^\infty \frac{dt}{r(t)} = \infty \text{ and } \lim_{x \rightarrow \infty} \left(\int_a^x \frac{dt}{r(t)} \right)^2 p(x) r(x) = 0.$$

Now we switch to the sublinear equations. Our results are again generalizations of Theorem 2.21 of Erbe. The methods used here are similar to those for the corresponding parts in Theorem 3.9.

THEOREM 3.11. Let $0 < \gamma < 1$. Then equation (3.9) is nonoscillatory in case any of the following hold:

$$(a) \int_a^\infty \left(\int_a^t \frac{dr}{r(r)} \right)^\gamma p(t) dt < \infty \text{ and}$$

$$\lim_{x \rightarrow \infty} (r(x)p(x))^{\frac{1-\gamma}{\gamma}} (Q_+(x))^{\frac{1-\gamma}{\gamma}} \int_x^\infty \left(\int_a^t \frac{dr}{r(r)} \right)^\gamma p(t) dt = 0;$$

$$(b) \int_a^\infty \left(\int_a^t \frac{dr}{r(r)} \right)^\gamma p(t) dt < \infty; \text{ and}$$

$$\lim_{x \rightarrow \infty} (Q_-(x))^{\frac{1-\gamma}{\gamma}} \int_x^\infty \left(\int_a^t \frac{dr}{r(r)} \right)^\gamma p(t) dt = 0;$$

$$(c) \int_a^\infty (p(t))^{\frac{1}{1-\gamma}} \left(\frac{Q_+(t)}{r(t)} \right)^{\frac{\gamma}{1-\gamma}} dt < \infty \text{ and}$$

$$\lim_{x \rightarrow \infty} (Q_-(x))^{\frac{1}{1-\gamma}} (Q_+(x))^{\frac{1}{1-\gamma}} \int_x^\infty (p(t))^{\frac{1}{1-\gamma}} \left(\frac{Q_+(t)}{r(t)} \right)^{\frac{\gamma}{1-\gamma}} dt = 0;$$

$$(d) \int_a^\infty (p(t))^{\frac{1}{1-\gamma}} \left(\frac{Q_+(t)}{r(t)} \right)^{\frac{\gamma}{1-\gamma}} dt < \infty \text{ and}$$

$$\lim_{x \rightarrow \infty} (r(x)p(x))^{\frac{1}{1-\gamma}} (Q_+(x))^{\frac{1}{1-\gamma}} \int_x^\infty (p(t))^{\frac{1}{1-\gamma}} \left(\frac{Q_+(t)}{r(t)} \right)^{\frac{\gamma}{1-\gamma}} dt = 0.$$

Proof: (a) We assume that $y(x)$ is an oscillatory solution of (3.9). Then there exists a sequence $x_n \rightarrow \infty$ with $y(x_n) = 0$ and $y'(x_n) > 0$. We let x_n be the first zero of $y'(x)$ exceeding x_n , then as in Theorem 3.9 part (d)

$$\begin{aligned} (r(x_n)y'(x_n)) &= \int_{x_n}^{x_n} p(t)(y(t))^\gamma dt \\ &= \int_{x_n}^{x_n} p(t) \left(\int_{x_n}^t y'(s) ds \right)^\gamma dt \\ &= \int_{x_n}^{x_n} p(t) \left(\int_{x_n}^t \frac{r(x)y'(s)}{r(s)} ds \right)^\gamma dt \\ &\leq (r(x_n)y'(x_n))^\gamma \int_{x_n}^{x_n} p(t) \left(\int_{x_n}^t \frac{dr}{r(r)} \right)^\gamma dt. \end{aligned}$$

Since $r(x)y'(x)$ is decreasing on $[x_n, x_n]$,

$$(r(x_n)y'(x_n))^{1-\gamma} \leq \int_{x_n}^{\infty} p(t) \left(\int_{x_n}^t \frac{d\tau}{r(\tau)} \right)^{\gamma} dt.$$

By Lemma 3.7,

$$\begin{aligned} (r(x_n)y'(x_n))^2 &= 2E(x_n) = 2(rp(x_n))B(x_n) \\ &\geq C(rp)(x_n)[Q_+(x_n)]^{-1}, \end{aligned}$$

where C is a positive constant and therefore,

$$1 \leq C_1 (r(x_n)p(x_n))^{\frac{1-\gamma}{2}} (Q_+(x))^{-\frac{1-\gamma}{2}} \int_{x_n}^{\infty} \left(\int_{x_n}^t \frac{d\tau}{r(\tau)} \right)^{\gamma} p(t) dt.$$

This is a contradiction.

(b) It is proved by part (a) and the relation (*).

(c) We again suppose that $y(x)$ is an oscillatory solution of (3.9) and $x_n \rightarrow \infty$ is a sequence with $y(x_n) = 0, y'(x_n) > 0$. Also x_n is the first zero of $y'(x)$ exceeding x_n . Then as in Theorem 3.9 (e), we have for $x > x_n$,

$$|y(x)|^{\gamma} \leq C \cdot (r(x_n)y'(x_n))^{\frac{\gamma}{1-\gamma}} (Q_+(x))^{\frac{1-\gamma}{1-\gamma}} (Q_+(x_n))^{-\frac{1-\gamma}{1-\gamma}} (rp(x))^{-\frac{1-\gamma}{1-\gamma}}.$$

Thus

$$\begin{aligned} r(x_n)y'(x_n) &= \int_{x_n}^{x_n} p(t)y^{\gamma}(t)dt \\ &\leq C(r(x_n)y'(x_n))^{\frac{\gamma}{1-\gamma}} (Q_+(x_n))^{\frac{1-\gamma}{1-\gamma}} \int_{x_n}^{x_n} (rp(t))^{\frac{1-\gamma}{1-\gamma}} p(t)(Q_+(t))^{\frac{1-\gamma}{1-\gamma}} dt, \end{aligned}$$

so

$$(3.12) \quad (r(x_n)y'(x_n))^{\frac{1-\gamma}{1-\gamma}} \leq C(Q_+(x_n))^{\frac{1-\gamma}{1-\gamma}} \int_{x_n}^{\infty} (p(t))^{\frac{1-\gamma}{1-\gamma}} \left(\frac{Q_+(t)}{r(t)} \right)^{\frac{1-\gamma}{1-\gamma}} dt.$$

Again from Lemma 3.7, let $\xi = a$

$$(r(x_n)y'(x_n))^2 = 2E(x_n) \geq 2E(a)Q_-(a)(Q_-(x_n))^{-1}.$$

This and (3.12) show that

$$1 \leq C_1(Q_-(x_n))^{\frac{1}{\gamma+1}}(Q_+(x_n))^{-\frac{1}{\gamma+1}} \int_{x_n}^{\infty} (p(t))^{\frac{1}{\gamma+1}} \left(\frac{Q_+(t)}{r(t)}\right)^{-\frac{1}{\gamma+1}} dt$$

and this is a contradiction.

(d) This can be proved by part (c) and the relation (*).

Next we combine the techniques used in this section and the previous section to obtain some more nonoscillation criteria. We need to assume that $r(x)$ is differentiable and $r(x) - 2xr'(x) > 0$. These are also analogues of results in Erbe [17]. When $r(x) \equiv 1$, they coincide with Theorem 3.9 parts (f) and (g).

THEOREM 3.12. *Let $\gamma > 1$. If $r(x)$ is differentiable and $r(x) - 2xr'(x) > 0$, then (3.9) is nonoscillatory in case any of the following hold:*

- (a) $\lim_{x \rightarrow \infty} \frac{x^{\gamma+1} p(x)}{r(x) - 2xr'(x)} \cdot \left[\frac{Q_+(x)}{r(x)(r(x) - \theta xr'(x))} \right]^{\frac{\gamma-1}{2}} = 0$ for some θ , $0 \leq \theta \leq 2$;
 (b) $\lim_{x \rightarrow \infty} \frac{x^{\gamma+1} p(x)^{\frac{\gamma+1}{2}}}{r(x) - 2xr'(x)} \cdot \left[\frac{Q_-(x)}{r(x)(r(x) - \theta xr'(x))} \right]^{\frac{\gamma-1}{2}} = 0$ for some θ , $0 \leq \theta \leq 2$.

Proof: (a) We make the change of variables

$$x = e^t, \quad y = e^{\frac{1}{2}t} u,$$

then equation (3.9) becomes

$$(\rho(t)u')' + \varphi(t)u^{\gamma} - \left(\frac{1}{4}\rho(t) - \frac{1}{2}\rho'(t)\right)u = 0$$

where $\rho(t) = r(e^t)$, $\varphi(t) = e^{\frac{\gamma+1}{2}t} p(e^t)$. Suppose $t_n \rightarrow \infty$ is a sequence of zeros of u with $u'(t_n) > 0$. Let τ_n be the first zero of $u'(t)$ exceeding t_n , then $u''(\tau_n) \leq 0$ and hence

$$(\rho(\tau_n)u'(\tau_n))' \leq 0,$$

therefore,

$$\varphi(\tau_n)(u(\tau_n))^\gamma - \left(\frac{1}{4}\rho(\tau_n) - \frac{1}{2}\rho'(\tau_n)\right)u(\tau_n) \geq 0.$$

So

$$(3.13) \quad |u(\tau_n)| \geq \left[\frac{\frac{1}{4}\rho'(\tau_n) - \frac{1}{2}\rho(\tau_n)}{\varphi(\tau_n)} \right]^{\frac{1}{\gamma+1}}.$$

Define the energy function

$$\begin{aligned} E(x) &= \frac{1}{2}(r(x)y'(x))^2 + \frac{rp(x)y^{\gamma+1}}{\gamma+1} \\ &= \frac{1}{2}(\rho(t))^2 \left(\frac{1}{2}u + u' \right)^2 e^{-t} + \frac{1}{\gamma+1} \gamma(t) \varphi(t) e^{-t} u^{\gamma+1} \end{aligned}$$

as in section 3.1. By (3.13),

$$\begin{aligned} E(\tau_n) &= e^{-\tau_n} \left[\frac{1}{8}(\rho(\tau_n)u(\tau_n))^2 + \frac{1}{\gamma+1} \rho(\tau_n) \varphi(\tau_n) u^{\gamma+1}(\tau_n) \right] \\ &\geq e^{-\tau_n} \left[\frac{\rho^2(\tau_n)}{8} \left(\frac{\frac{1}{4}\rho(\tau_n) - \frac{1}{2}\rho'(\tau_n)}{\varphi(\tau_n)} \right)^{\frac{2}{\gamma+1}} \right. \\ &\quad \left. + \frac{1}{\gamma+1} \rho(\tau_n) \varphi(\tau_n) \left(\frac{\frac{1}{4}\rho(\tau_n) - \frac{1}{2}\rho'(\tau_n)}{\varphi(\tau_n)} \right)^{\frac{\gamma+1}{\gamma+1}} \right] \\ &= e^{-\tau_n} \left[\frac{\rho(\tau_n) - 2\rho'(\tau_n)}{\varphi(\tau_n)} \right]^{\frac{2}{\gamma+1}} \\ &\quad \left[\frac{1}{8} \left(\frac{1}{4} \right)^{\frac{2}{\gamma+1}} \rho^2(\tau_n) + \frac{1}{\gamma+1} \left(\frac{1}{4} \right)^{\frac{\gamma+1}{\gamma+1}} \rho(\tau_n) (\rho(\tau_n) - \rho'(\tau_n)) \right] \\ &\geq \rho(\tau_n) e^{-\tau_n} \left[\frac{\rho(\tau_n) - 2\rho'(\tau_n)}{\varphi(\tau_n)} \right]^{\frac{2}{\gamma+1}} [C_1 \rho(\tau_n) + C_2 (\rho(\tau_n) - \rho'(\tau_n))] \end{aligned}$$

where C_1 and C_2 can be chosen so that C_1/C_2 equals to any nonnegative number of infinity. Thus for any θ , $0 \leq \theta \leq 2$, there is a constant C such that

$$E(\tau_n) \geq C\gamma(\tau_n)e^{-\tau_n} \left[\frac{\rho(\tau_n) - 2\rho'(\tau_n)}{\varphi(\tau_n)} \right]^{\frac{2}{\gamma-1}} (\rho(\tau_n) - \theta\rho'(\tau_n))$$

From Lemma 3.7, we have

$$E(\tau_n) \leq C_1 Q_+(e^{\tau_n}) \quad (\xi = a).$$

Thus

$$\begin{aligned} 1 &\leq C Q_+(e^{\tau_n}) \frac{e^{\tau_n}}{p(\tau_n)} \left(\frac{\phi(\tau_n)}{\rho(\tau_n) - 2\rho'(\tau_n)} \right)^{\frac{2}{\gamma-1}} \frac{1}{\rho(\tau_n) - \theta\rho'(\tau_n)} \\ &= C Q_+(e^{\tau_n} e^{\frac{2\gamma+1}{\gamma-1}\tau_n}) \left(\frac{p(e^{\tau_n})}{\rho(\tau_n) - 2\rho'(\tau_n)} \right)^{\frac{2}{\gamma-1}} \frac{1}{\rho(\tau_n) - \theta\rho'(\tau_n)}. \end{aligned}$$

Since $\rho'(t) = x \frac{d}{dx} r(x)$ we obtain a contradiction.

(b) This is from part (a) and the relation (*).

We also have the sublinear version for this theorem.

THEOREM 3.13. Let $0 < \gamma < 1$. if $r(x)$ is differentiable and $r(x) - 2xr'(x) > 0$, then (3.9) is nonoscillatory in case any of the following hold:

- (a) $\lim_{x \rightarrow \infty} [Q_-(x)r(x)(r(x) - \theta xr'(x))]^{\frac{1-\gamma}{2}} \frac{x^{1+\gamma} p(x)}{r(x) - 2xr'(x)} = 0$, for some $0 < \theta < 2$;
- (b) $\lim_{x \rightarrow \infty} [Q_+(x)(r(x) - \theta xr'(x))]^{\frac{1-\gamma}{2}} \frac{(x^2 p(x))^{\frac{1+\gamma}{2}}}{r(x) - 2xr'(x)} = 0$, for some $0 < \theta < 2$.

Proof: (a) As in Theorem 3.12. We make the change of variables,

$$x = e^t, \quad y = e^{\frac{1}{2}t} u.$$

Equation (3.9) becomes

$$(\rho(t)u')' + \varphi(t)u^\gamma - \left(\frac{1}{4}\rho(t) - \frac{1}{2}\rho'(t)\right)u = 0.$$

where $\rho(t) = r(e^t)$, and $\varphi(t) = P(e^t)e^{\frac{\gamma+3}{2}t}$. We choose $t_n \rightarrow \infty$ and τ_n as before.

Now we have

$$|u(\tau_n)| \leq \left[\frac{\phi(\tau_n)}{\frac{1}{4}\rho(\tau_n) - \frac{1}{2}\rho'(\tau_n)} \right]^{\frac{1}{\gamma-1}}$$

and

$$\begin{aligned} E(\tau_n) &= e^{-\tau_n} \left[\frac{1}{8}(\rho(\tau_n)u(\tau_n))^2 + \frac{1}{\gamma+1}\rho(\tau_n)\varphi(\tau_n)u^{\gamma+1}(\tau_n) \right] \\ &\geq e^{-\tau_n} \left[\frac{1}{8}\rho(\tau_n) \left(\frac{\varphi(\tau_n)}{\frac{1}{4}\rho(\tau_n) - \frac{1}{2}\rho'(\tau_n)} \right)^{\frac{2}{\gamma-1}} + \frac{1}{\gamma+1}\rho(\tau_n)\varphi(\tau_n) \left(\frac{\varphi(\tau_n)}{\frac{1}{4}\rho(\tau_n) - \frac{1}{2}\rho'(\tau_n)} \right)^{\frac{\gamma+1}{\gamma-1}} \right] \\ &= C\rho(\tau_n)e^{-\tau_n} \left[\frac{\varphi(\tau_n)}{\frac{1}{4}\rho(\tau_n) - \frac{1}{2}\rho'(\tau_n)} \right]^{\frac{2}{\gamma-1}} (\rho(\tau_n) - \theta\rho'(\tau_n)) \end{aligned}$$

for any $0 < \theta < 2$ and $c > 0$ as we have shown in the previous theorem. From

Lemma 3.7, we have

$$E(\tau_n) \geq c_1[Q_-(e^{\tau_n})]^{-1}$$

and

$$\begin{aligned} 1 &\leq CQ_-(e^{\tau_n})\rho(\tau_n)e^{-\tau_n} \left[\frac{\varphi(\tau_n)}{\rho(\tau_n) - 2\rho'(\tau_n)} \right]^{\frac{2}{\gamma-1}} [\rho(\tau_n) - \theta\rho'(\tau_n)] \\ &= CQ_-(e^{\tau_n})e^{\frac{2(1+\gamma)}{1-\gamma}\tau_n} \left[\frac{\rho(e^{\tau_n})}{\rho(\tau_n) - 2\rho'(\tau_n)} \right]^{\frac{2}{\gamma-1}} \rho(\tau_n)[\rho(\tau_n) - \theta\rho'(\tau_n)] \end{aligned}$$

again this is a contradiction.

By the equality (*), one can prove part (b).

§3 Conditions For Oscillatory Solution to Exist

In section 3.1 and 3.2 we obtained conditions under which the equation

$$(3.14) \quad (r(x)y'(x))' + p(x)y^\gamma(x) = 0, \quad \gamma > 0$$

is nonoscillatory. This section is devoted to the condition for (3.14) to have oscillatory solutions. The criteria we are going to establish are improvements of Theorem 2.12 and 2.19 of Erbe and Muldowney [19].

At first we state two theorems of Erbe and Muldowney [19] which we shall use in the sequel. Consider the more general equation

$$(3.15) \quad (r(t)u'(t))' + a(t, u)u = 0, \quad t \geq a$$

where $r(t)$ is positive and such that $\int_a^\infty \frac{dt}{r(t)} = \infty$.

THEOREM 3.14. (Erbe and Muldowney). Suppose $a(t, u) = a(t_1 - u)$ for each $(t, u) \in (a, \infty) \times R$. Equation (3.15) has an oscillatory solution provided there exist real valued functions $\pi(t, u)$, $\nu(t, u)$, $\alpha(t)$, $\mu(t_0, t)$ and a constant k such that

- (i) $r(t)a(t, u) = \pi(t, u) - \nu(t, u)$;
- (ii) $\pi(t, u), \nu(t, u)$ are nondecreasing in t for each u ;
- (iii) $|u| > \alpha(t) \Leftrightarrow 2 \int_0^u [\pi(t, s) - \nu(\infty, s)] s ds > k \geq 0$;
- (iv) $|u| > \alpha(t_0), t \geq t_0 \Rightarrow a(t, u) \geq \mu(t_0, t) \geq 0$ and $(r(t)u'(t))' + \mu(t_0, t)u = 0$

is oscillatory.

THEOREM 3.15. Suppose there exists a neighbourhood $U_0 = U(t_0)$ of the half-line $\{(t, u) : t \in (t_0, \infty), u = 0\}$, a constant $\delta_0 = \delta(t_0) > 0$ and real valued functions $\mu_0(t) = \mu(t_0, t)$, $K_0(t) = K(t_0, t)$, $t \in [t_0, \infty)$. If

- (i) $(t, u) \in U_0 \Rightarrow a(t, u) \geq \mu_0(t)$ where $(ru')' + \mu_0(t_0, t)u = 0$ is oscillatory;
 (ii) $(t, u) \in U_0 \Rightarrow K_0(t, u)$ is nonincreasing in t ;
 (iii) $(t, u) \in \partial U_0 \Rightarrow K_0(t, u) \geq \delta(t_0)$ for each $t \geq t_0$, where $K_0(t, u) = K_0(t_0, t, u) = 2r(t) \int_0^u a(t, s) s ds - K_0(t)$,
 then any solution of (3.15) such that $(t_0, u(t_0)) \in U_0$, $K_0(t_0, u(t_0)) + (r(t_0)u'(t_0))^2 < \delta_0$ is oscillatory.

Now we consider the superlinear case of our problem. By the change of variables

$$x = e^t, y = e^{(\mu + \frac{1}{2})t} u, -\frac{1}{2} \leq \mu \leq 0$$

as in section 3.1, (3.14) becomes

$$(3.16) \quad [e^{2\mu t} \rho(t) u']' + a(t, u)u = 0$$

where $a(t, u) = e^{2\mu t} [\sigma(t)u^{\gamma+1} - \lambda(t)]$, and,

$$\sigma(t) = p(e^t) e^{(\gamma(\mu + \frac{1}{2}) + \frac{3}{2} - \mu)t}$$

and

$$\lambda(t) = (\frac{1}{4} - \mu^2)\rho(t) - (\frac{1}{2} + \mu)\rho'(t), \text{ here } \rho(t) = r(e^t).$$

THEOREM 3.16. Let $\gamma > 1$. Suppose for some real number $-\frac{1}{2} \leq \mu \leq 0$, and

- (i) $r(x)p(x)x^\delta = \hat{\pi}(x) - \hat{\nu}(x)$, where $\hat{\pi}, \hat{\nu}$ are nondecreasing $\hat{\pi}(x) - \hat{\nu}(\infty) > 0$
 and $\delta = \mu(\gamma + 3) + \frac{\gamma+3}{2}$ or $(\gamma + 3)(\frac{1}{2} + \mu)$;
 (ii) $x^{4\mu}[(\frac{1}{4} - \mu^2)r^2(x) - (\mu + \frac{1}{2})xr(x)r'(x)]$ is nonnegative and nonincreasing;
 (iii) $\int^\infty \frac{ds}{r(s)s^{\frac{\gamma+1}{2}+1}} = \infty$;

then Equation (3.14) has an oscillatory solution.

Proof: We calculate that

$$\begin{aligned}
 e^{2\mu t} \rho(t) a(t, u) &= e^{2\mu t} \rho(t) e^{2\mu t} (\sigma(t) u^{\gamma-1} - \lambda(t)) \\
 &= e^{4\mu t} \rho(t) [p(e^t) e^{\gamma(\mu+\frac{1}{2})+\frac{3}{2}-\mu)t} u^{\gamma-1} - \lambda(t)] \\
 &= \rho(t) p(e^t) e^{\gamma(\mu+\frac{1}{2})+\frac{3}{2}+3\mu)t} u^{\gamma-1} - e^{4\mu t} \lambda(t) \rho(t) \\
 &= (\hat{\pi}(e^t) - \hat{\nu}(e^t)) u^{\gamma-1} - e^{4\mu t} \rho(t) \lambda(t) \\
 &= \hat{\pi}(e^t) u^{\gamma-1} - e^{4\mu t} \rho(t) \lambda(t) - \hat{\nu}(e^t) u^{\gamma-1}.
 \end{aligned}$$

We denote

$$\pi(t, u) = \hat{\pi}(e^t) u^{\gamma-1} - e^{4\mu t} \rho(t) \lambda(t),$$

$$\nu(t, u) = \hat{\nu}(e^t) u^{\gamma-1}.$$

By condition (ii) and (i) $\pi(t, u)$ and $\nu(t, u)$ are nondecreasing in t for a fixed u .

Let

$$\alpha(t) = \left[\frac{\gamma+1}{2} \cdot \frac{e^{4\mu t} \rho(t) \lambda(t)}{\hat{\pi}(e^t) - \hat{\nu}(\infty)} \right]^{\frac{1}{\gamma-1}}.$$

Since

$$\begin{aligned}
 \int_0^u (\pi(t, s) - \nu(\infty, s)) s ds &= \int_0^u (\hat{\pi}(e^t) s^{\gamma-1} - e^{4\mu t} \rho(t) \lambda(t) - \hat{\nu}(\infty) s^{\gamma-1}) s ds \\
 &= \frac{\hat{\pi}(e^t) - \hat{\nu}(\infty)}{\gamma+1} u^{\gamma+1} - \frac{e^{4\mu t} \rho(t) \lambda(t)}{2} u^2,
 \end{aligned}$$

we have

$$\int_0^u [\pi(t, s) - \nu(\infty, s)] s ds > 0$$

if and only if

$$|u| > \alpha(t) = \left[\frac{\gamma + 1}{2} \frac{e^{4\mu t} \rho(t) \lambda(t)}{\hat{\pi}(e^t) - \hat{\nu}(\infty)} \right]^{\frac{1}{\gamma-1}}.$$

Furthermore, let

$$\mu(t_0, t) = \left(\frac{\gamma - 1}{2} \right) \frac{e^{4\mu t_0} \lambda(t_0) \rho(t_0)}{\rho(t) e^{2\mu t}},$$

then if $t \geq t_0$ and $|u| \geq \alpha(t_0)$, we have

$$\begin{aligned} a(t, u) &= e^{2\mu t} (\sigma(t) u^{\gamma-1} - \lambda(t)) \\ &\geq e^{2\mu t} \left[\frac{\sigma(t)(\gamma + 1) e^{4\mu t_0} \rho(t_0) \lambda(t_0)}{2(\hat{\pi}(e^{t_0}) - \hat{\nu}(\infty))} - \lambda(t) \right] \\ &= \frac{e^{4\mu t} \sigma(t)(\gamma + 1) e^{4\mu t_0} \rho(t_0) \lambda(t_0)}{2e^{2\mu t} (\hat{\pi}(e^{t_0}) - \hat{\nu}(\infty))} - \lambda(t) e^{2\mu t} \\ &= \frac{(\gamma + 1) [\hat{\pi}(e^t) - \hat{\nu}(e^t)] e^{4\mu t_0} \rho(t_0) \lambda(t_0)}{2\rho(t) e^{2\mu t} (\hat{\pi}(e^{t_0}) - \hat{\nu}(\infty))} - \lambda(t) e^{2\mu t} \\ &\geq \frac{(\gamma + 1) e^{4\mu t_0} \rho(t_0) \lambda(t_0) - 2\rho(t) \lambda(t) e^{4\mu t}}{2\rho(t) e^{2\mu t}} \\ &\geq \frac{(\gamma + 1) e^{4\mu t_0} \lambda(t_0) \rho(t_0) - 2\rho(t_0) \lambda(t_0) e^{4\mu t_0}}{2\rho(t_0) e^{2\mu t}} \\ &= \frac{(\gamma - 1) e^{4\mu t_0} \lambda(t_0) \rho(t_0)}{2\rho(t) e^{2\mu t}} = \mu(t_0, t). \end{aligned}$$

Consider the equation

$$(3.17) \quad (\rho(t) e^{2\mu t} u')' + \mu(t_0, t) u = 0.$$

Since $\int_0^\infty \frac{dt}{\rho(t) e^{2\mu t}} = \infty$, by Theorem 2.7 of Leighton, equation (3.17) is oscillatory.

Then by Theorem 3.14, equation (3.16) has an oscillatory solution.

COROLLARY 3.17. Suppose

(i) $r(x)p(x)x^{\frac{\gamma+3}{2}} = \hat{\pi}(x) + \hat{\nu}(x)$, where $\hat{\pi}, \hat{\nu}$ are nondecreasing and $\hat{\pi}(x) - \hat{\nu}(\infty) > 0$;

(ii) $\frac{1}{4}r^2(x) - \frac{1}{2}xr(x)r'(x)$ is nonnegative and nonincreasing;

(iii) $\int^{\infty} \frac{dx}{xr(x)} = \infty$;

then equation (3.14) has an oscillatory solution.

Proof: Put $\mu = 0$ in the theorem.

COROLLARY 3.18. Suppose

(i) $r(x)p(x) = \hat{\pi}(x) - \hat{\nu}(x)$, where $\hat{\pi}(x), \hat{\nu}(x)$ are nondecreasing and $\hat{\pi}(x) - \hat{\nu}(\infty) > 0$;

(ii) $\int^{\infty} \frac{dx}{r(x)} = \infty$;

then equation (3.14) has an oscillatory solution.

Proof: Put $\mu = -\frac{1}{2}$ in the theorem.

COROLLARY 3.19. Suppose $p(x)x^{\delta} = \hat{\pi}(x) - \hat{\nu}(x)$, where $\hat{\pi}(x), \hat{\nu}(x)$ are nondecreasing and $\hat{\pi}(x) - \hat{\nu}(\infty) > 0$, and $\delta = \mu(\gamma + 3) + \frac{\gamma+3}{2}$, then the equation

$$y'' + p(x)y^{\gamma} = 0$$

has an oscillatory solution.

Corollary 3.19 is the well-known result of Erbe and Muldowney. It is now the special case of our theorem when $r(x) \equiv 1$.

COROLLARY 3.20. Suppose $r(x)$ satisfies conditions (ii) and (iii) of Corollary 3.17. If $r(x)p(x)x^{\frac{\gamma+1}{2}}$ is nondecreasing, then (3.14) has an oscillatory solution.

COROLLARY 3.21. Suppose $r(x)$ satisfies conditions (ii) and (iii) of Corollary 3.17. If $r(x)p(x)x^{\frac{\gamma+3}{2}}$ is nonincreasing and bounded below, then (3.14) has an oscillatory solution.

COROLLARY 3.22. Suppose $\int_a^\infty \frac{dx}{r(x)} = \infty$. If $r(x)p(x)$ is nondecreasing or nonincreasing and bounded below, then (3.14) has an oscillatory solution.

For the sublinear equation we have similar results.

THEOREM 3.23. Let $0 < \gamma < 1$. Suppose for some $-\frac{1}{2} \leq \mu \leq 0$.

(i) $r(x)p(x)x^\delta = \hat{\pi}(x) - \hat{\nu}(x)$, $\hat{\pi}(x)$, $\hat{\nu}(x)$ are nondecreasing and $\hat{\pi}(x) - \hat{\nu}(\infty) > 0$, where $\delta = (\gamma + 3)(\frac{1}{2} + \mu)$;

(ii) $\int_a^\infty \frac{dx}{r(x)x^{4\mu+1}} = \infty$;

(iii) $x^{4\mu}[(\frac{1}{4} - \mu^2)r^2(x) - (\mu + \frac{1}{2})xr(x)r'(x)]$ is nonnegative and nonincreasing then equation (3.14) has oscillatory solutions.

Proof: Let $x = e^t$, $y = e^{(\frac{1}{2} + \mu)t} u$. Then equation (3.14) becomes

$$(e^{2\mu t} \rho(t) u')' + a(t, u) u = 0$$

where $a(t, u) = e^{2\mu t} (\sigma(t) u^{\gamma-1} - \lambda(t))$, $\sigma(t) = p(e^t) e^{(\gamma(\mu + \frac{1}{2}) + \frac{3}{2} - \mu)t}$ and

$$\lambda(t) = (\frac{1}{4} - \mu^2) \rho(t) - (\mu + \frac{1}{2}) \rho'(t), \quad \rho(t) = r(e^t)$$

Note that

$$e^{2\mu t} \rho(t) a(t, u) = [\hat{\pi}(e^t) - \hat{\nu}(e^t)] u^{\gamma-1} - e^{4\mu t} \rho(t) \lambda(t)$$

and $a(t, u) > 0$ if and only if

$$|u| < \left[\frac{\rho(t)\lambda(t)e^{4\mu t}}{\widehat{\pi}(e^t) - \widehat{\nu}(e^t)} \right]^{\frac{1}{\gamma-1}}$$

Let

$$\alpha(t) = \left[\frac{\rho(t)\lambda(t)e^{4\mu t}}{c(\widehat{\pi}(e^t) - \widehat{\nu}(\infty))} \right]^{\frac{1}{\gamma-1}}, 0 < c < 1$$

and

$$U_0 = \{(t, u) : t \in [t_0, \infty), |u| < \alpha(t_0)\}.$$

Then $(t, u) \in U_0$ implies that

$$\begin{aligned} a(t, u) &= \frac{[\widehat{\pi}(e^t) - \widehat{\nu}(e^t)]u^{\gamma-1} - \lambda(t)\rho(t)e^{2\mu t}}{\rho(t)e^{2\mu t}} \\ &\geq \frac{\frac{1}{c}e^{4\mu t}\rho(t_0)\lambda(t_0) - \lambda(t)\rho(t)e^{4\mu t}}{\rho(t)e^{2\mu t}} \\ &\geq \frac{\frac{1}{c}e^{4\mu t_0}\rho(t_0)\lambda(t_0) - \lambda(t_0)\rho(t_0)e^{4\mu t_0}}{\rho(t)e^{2\mu t}} \\ &= \frac{e^{4\mu t_0}\rho(t_0)\lambda(t_0)(\frac{1}{c} - 1)}{\rho(t)e^{2\mu t}}. \end{aligned}$$

we define

$$\mu(t_0, t) = e^{4\mu t_0}\rho(t_0)\lambda(t_0)(\frac{1}{c} - 1)/\rho(t)e^{2\mu t}.$$

By condition (ii) and Theorem 2.7 of Leighton, the equation

$$(\rho(t)e^{2\mu t}u')' + \mu(t_0, t)u = 0$$

is oscillatory.

Let

$$K(t, u) = 2\rho(t)e^{2\mu t} \int_0^u a(t, s)sd s - \frac{2}{\gamma+1}\alpha^{\gamma+1}(t_0) \int_{t_0}^t d\widehat{\pi} + \alpha^2(t_0) \int_{t_0}^t d(\rho(s)e^{4\mu s}\lambda(s))$$

for $(t, u) \in U_0$. If $t_2 > t_1$,

$$\begin{aligned} k(t_2, u) - K(t_1, u) &= \frac{2u^{\gamma+1}}{\gamma+1} \int_{t_1}^{t_2} d\hat{\pi} - u^2 \int_{t_1}^{t_2} d(\rho(s)e^{4\mu s} \lambda(s)) \\ &\quad + \alpha^2(t_0) \int_{t_1}^{t_2} d(\rho(s)\lambda(s)e^{4\mu s}) - \frac{2\alpha^{\gamma+1}(t_0)}{\gamma+1} \int_{t_1}^{t_2} d\hat{\pi} \leq 0. \end{aligned}$$

Hence $k(t, u)$ is nonincreasing in t . Finally

$$\begin{aligned} K(t, \pm\alpha(t_0)) &= 2\rho(t)e^{4\mu t} \left[\sigma(t) \frac{\alpha^{\gamma+1}(t_0)}{\gamma+1} - \frac{\lambda(t)\alpha^2(t_0)}{2} \right] - \frac{2}{\gamma+1} \alpha^{\gamma+1}(t_0) \int_{t_0}^t d\hat{\pi} \\ &\quad + \alpha^2(t_0) \int_{t_0}^t d(\rho(s)e^{4\mu s} \lambda(s)) \\ &= \frac{2}{\gamma+1} \alpha^{\gamma+1}(t_0) [\hat{\pi}(e^t) - \hat{\nu}(e^t)] - \alpha^2(t_0) \rho(t) \lambda(t) e^{4\mu t} \\ &\quad - \frac{2}{\gamma+1} \alpha^{\gamma+1}(t_0) [\hat{\pi}(t) - \hat{\pi}(t_0)] \\ &\quad + \alpha^2(t_0) (\rho(t)e^{4\mu t} \lambda(t) - \rho(t_0)e^{4\mu t_0} \lambda(t_0)) \\ &= \frac{2}{\gamma+1} \alpha^{\gamma+1}(t_0) [\hat{\pi}(e^{t_0}) - \hat{\nu}(e^t)] - \alpha^2(t_0) \rho(t_0) \lambda(t_0) e^{4\mu t_0} \\ &\geq \frac{2}{\gamma+1} \alpha^{\gamma+1}(t_0) [\hat{\pi}(e^{t_0}) - \hat{\nu}(\infty)] - \alpha^2(t_0) \rho(t_0) \lambda(t_0) e^{4\mu t_0} \\ &= \alpha^2(t_0) \left[\frac{2}{\gamma+1} \alpha^{\gamma-1}(t_0) (\hat{\pi}(e^{t_0}) - \hat{\nu}(\infty)) - \lambda(t_0) \rho(t_0) e^{4\mu t_0} \right] \\ &= \alpha^2(t_0) \left[\frac{2}{c(\gamma+1)} - 1 \right] \lambda(t_0) \rho(t_0) e^{4\mu t_0} \\ &= \delta(t_0) > 0. \end{aligned}$$

Since $\frac{2}{c(\gamma+1)} > 1$, by Theorem 3.15, equation (3.14) has oscillatory solutions.

We also have analogues of Corollaries 3.17 - 3.22 for the sublinear equation.

COROLLARY 3.24. Suppose $0 < \gamma < 1$ and

(i) $r(x)p(x)x^{\frac{\gamma+1}{2}} = \hat{\pi}(x) - \hat{\nu}(x)$, where $\hat{\pi}, \hat{\nu}$ are nondecreasing and $\hat{\pi}(x) -$

$\hat{\nu}(\infty) > 0$;

(ii) $\frac{1}{4}r^2(x) - \frac{1}{2}xr(x)r'(x)$ is nonnegative and nonincreasing;

(iii) $\int_a^\infty \frac{dx}{xr(x)} = \infty$.

Then equation (3.14) has an oscillatory solution.

COROLLARY 3.25. Suppose $0 < \gamma < 1$ and

(i) $r(x)p(x) = \hat{\pi}(x) - \hat{\nu}(x)$, where $\hat{\pi}(x)$, $\hat{\nu}(x)$ are nondecreasing and $\hat{\pi}(x) - \hat{\nu}(\infty) > 0$;

(ii) $\int_a^\infty \frac{dx}{r(x)} = \infty$;

then equation (3.14) has an oscillatory solution.

COROLLARY 3.26. Suppose $r(x)$ satisfies condition (ii) and (iii) of Corollary

3.17. If $r(x)p(x)x^{\frac{\gamma+1}{2}}$ is nondecreasing, then (3.14) has an oscillatory solution.

COROLLARY 3.27. Suppose $r(x)$ satisfies condition (ii) and (iii) of Corollary

3.17. If $r(x)p(x)x^{\frac{\gamma+1}{2}}$ is nonincreasing and bounded below, then (3.14) has an oscillatory solution.

COROLLARY 3.28. Suppose $\int_a^\infty \frac{dx}{r(x)} = \infty$. If $r(x)p(x)$ is nondecreasing or nonincreasing and bounded below, then (3.14) has an oscillatory solution.

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